

High Temperature Series

Systematic expansion
in powers of $(J/k_B T)$

First example: Ising chain

$$H = -J \sum_{j=1}^{N-1} \sigma_j \sigma_{j+1} \quad Z = \sum_{\vec{\sigma}} e^{-\beta H} \quad \beta \text{ small}$$

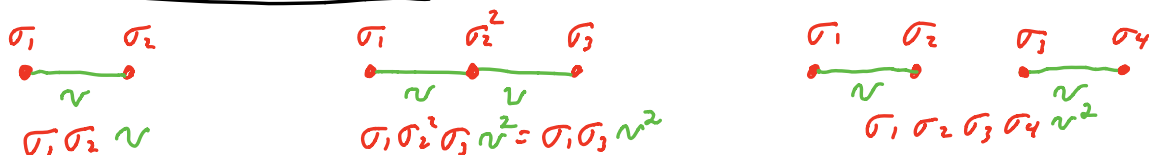
open boundary, $h=0$

Note: $\sigma \sigma' = \pm 1 \Rightarrow e^{\beta J \sigma \sigma'} = e^{\pm \beta J} = \cosh(\beta J) + \sigma \sigma' \sinh(\beta J)$
 $= \cosh(\beta J) (1 + \sigma \sigma' \tanh(\beta J))$

$$Z = \sum_{\vec{\sigma}} \prod_{j=1}^{N-1} e^{\beta J \sigma_j \sigma_{j+1}} = \cosh^{N-1}(\beta J) \sum_{\vec{\sigma}} \prod_j (1 + \sigma_j \sigma_{j+1} v) \quad v \text{ small}$$

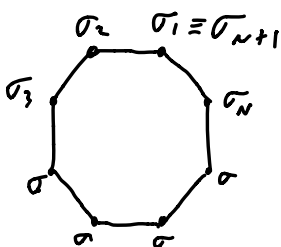
Expand $\prod_j (1 + \sigma_j \sigma_{j+1} v) = (1 + \sigma_1 \sigma_2 v)(1 + \sigma_2 \sigma_3 v) \dots (1 + \sigma_{N-1} \sigma_N v)$
 $= 1 + \sum_j \sigma_j \sigma_{j+1} v + \sum_j \sum_{k>j} \sigma_j \sigma_{j+1} \sigma_k \sigma_{k+1} v^2 + \dots$
 $\sum_{\vec{\sigma}} \rightarrow 2^N \quad \sum_{\vec{\sigma}} \rightarrow 0 \quad \dots \quad \sum_{\vec{\sigma}} \rightarrow 0$

Graphical Representation



Finally, $Z = 2^N \cosh^{N-1}(\beta J)$ as found on April 6 for free B.C.

Periodic B.C. $\prod_{j=1}^N e^{\beta J \sigma_j \sigma_{j+1}} = 1 + \sum_j \sigma_j \sigma_{j+1} v + \sum_{j,k>j} \sigma_j \sigma_{j+1} \sigma_k \sigma_{k+1} v^2 + \dots + \sum_{j<k<\dots<N} \sigma_j \sigma_{j+1} \sigma_{k+1} \dots \sigma_N \sigma_1 v^N$



$$Z = 2^N \cosh^N(\beta J) (1 + v^N)$$

$\rightarrow 2^N \cosh^N(\beta J)$ as $N \rightarrow \infty$ as found on April 7

Magnetic field: Note $e^{\beta h \sigma} = \cosh(\beta h) (1 + \underbrace{\sigma \tanh(\beta h)}_{\substack{u \quad \text{small}}})$

$$Z = (\cosh \beta J)^N (\cosh \beta h)^N \sum_{\vec{\sigma}} (1 + \sum_j \sigma_j \sigma_{j+1} v + \dots) (1 + \sum_k \sigma_k u + \sum_{k,l>k} \sigma_k \sigma_l u^2 + \dots)$$

$\sum_{\vec{2}} \text{vanishes except terms like } 1 \text{ and } \sigma_j \sigma_{j+1} \overset{N}{2} \overset{N}{2} \sigma_h \sigma_l \omega^2 \text{ with } h=j, l=j+1$

Graphical Representation



$$\therefore Z = 2^N \cosh^N(\beta J) \cosh^N(\beta h) (1 + N u^2 v + \dots)$$

$$M = - \frac{\partial}{\partial h} (-k_B T \ln Z) = N \sinh(\beta h) \cosh(\beta J) (1 + 2 \tanh(\beta J) \sinh(\beta h))$$

$$= N \tanh(\beta h) + 2 N \tanh(\beta J) \tanh(\beta h) + \mathcal{O}(h^3)$$

$$\chi_0 = \frac{1}{N} \left. \frac{\partial M}{\partial h} \right|_{h=0} = \frac{1}{k_B T} (1 + 2 \tanh(J/k_B T))$$

↑ $\begin{cases} \text{enhances } \pi & \text{if } J > 0 \\ \text{reduces } \pi & \text{if } J < 0 \end{cases}$