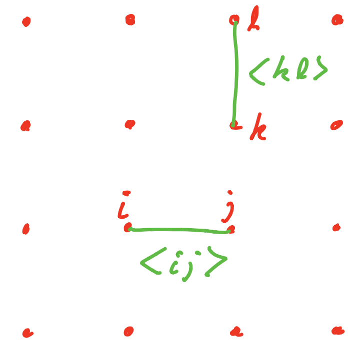
2D Square Ising high T expansion

$$H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j - h\sum_i \sigma_i$$

$N = L\times L$ sites

$2N$ bonds

2 bonds/spin



Set $h=0$

$$e^{\beta J\sigma\sigma'} = \cosh(\beta J)(1+\sigma\sigma' v) \qquad v = \tanh(\beta J)$$

$$Z = \sum_{\vec\sigma}\prod_{\langle ij\rangle} e^{\beta\sigma_i\sigma_j} = \cosh^{2N}(\beta J)\sum_{\vec\sigma}\Big(1+\underbrace{\sum_{\langle ij\rangle}}_{\text{all bonds}}\sigma_i\sigma_j v + \underbrace{\sum_{\langle ij\rangle \neq \langle kl\rangle}}_{\text{all pairs of different bonds}}\sigma_i\sigma_j\sigma_k\sigma_l v^2 + \cdots\Big)$$

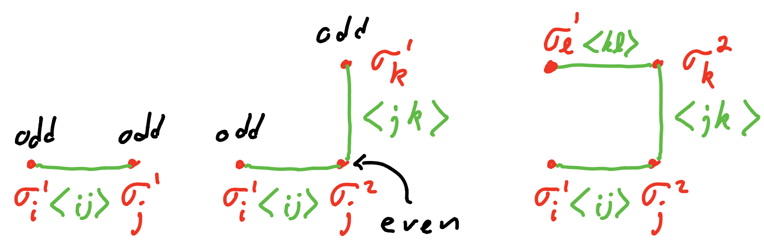
Generic term: $v^n\cdot \sum_{\vec\sigma}$ (n-tuplet of bonds)

$\uparrow \sigma_i^{n_i}\sigma_j^{n_j}\cdots\sigma_l^{n_l}$ with $n_i+n_j+\cdots+n_l = 2n$

Note: $$\sum_{\vec\sigma}\sigma_i^{n_i}\sigma_j^{n_j}\cdots\sigma_l^{n_l} = \begin{cases} 2^N & \text{if all } n_i, n_j, \cdots, n_l \text{ are even}\\ 0 & \text{else}\end{cases}$$

Terms surviving $\sum_{\vec\sigma}$ must have even # bonds/site

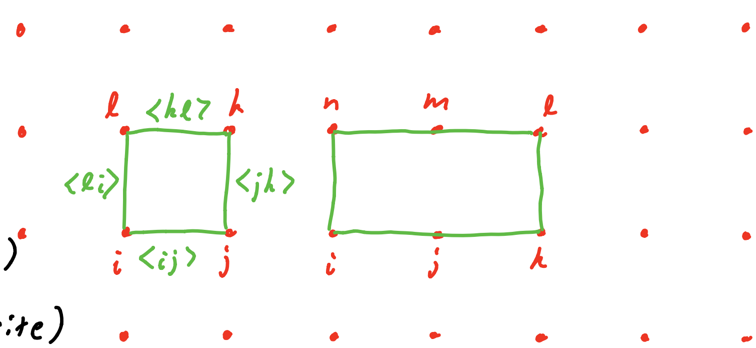
Graphical examples:
all sum to zero $\longrightarrow$



Closed loops:

$n=4$ square

$n=6$ rectangle



squares $= N$ (1 square/site)

rectangles $= 2N$ (1H+1V/site)

$$\therefore\ Z = 2^N\cosh^{2N}(\beta J)(1+Nv^4+2Nv^6+\cdots)$$

Dimension-dependence

1-D : First closed loop has N sites (PBC) $\Rightarrow Z = 2^N \cosh^N (1+v^N)$

2-D : First closed loop is square $\Rightarrow Z = 2^N \cosh^{2N} (1 + N v^4 + \cdots)$

$\parallel \hat{x}, \hat{y}$

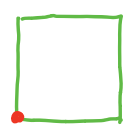
3-D : " $\Rightarrow Z = 2^N \cosh^{3N} (1 + 3 N v^4 + \cdots)$

$\vdots$

$\parallel \hat{x}, \hat{y}, \hat{z}$ xy, yz, zx

Extension to high order

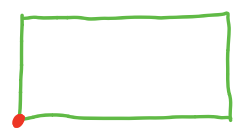
$n = 4$



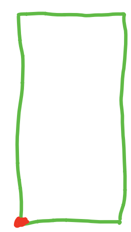
"root" $N\times$

N squares

$n = 6$



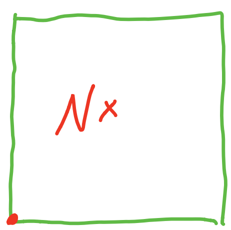
$N\times$ horizontal



$N\times$ vertical

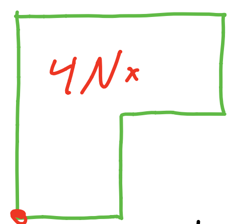
$2N$ rectangles

$n = 8$ connected

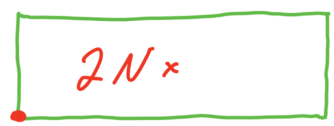


$N\times$

root



$4N\times$

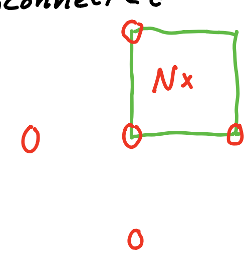


$2N\times$

$7N$ * Connected loops

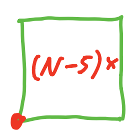
$\frac{1}{2} N(N-5)$ disconnected loops

$n=8$ disconnected



$N\times$

forbidden sites



$(N-5)\times$

root

$\frac{1}{2} N(N-5) + 7N = \frac{1}{2} N(N+9)$ terms total

$n = 10 \Rightarrow 2N(N+6)$ terms

$\boxed{\begin{array}{l} \cosh^2 - \sinh^2 = \cosh^2(1-\tanh^2) = 1 \\ \Rightarrow \cosh^2 = \frac{1}{1-\tanh^2} \leftarrow \text{useful fact!} \end{array}}$

$$\therefore Z = 2^N \cosh^{2N}(\beta J)\left(1 + N v^2 + 2N v^6 + \tfrac{1}{2} N(N+9) v^8 + 2N(N+6) v^{10} + \cdots\right)$$

$$F = -NkT \left\{ \ln 2 + v^2 + \tfrac{3}{2} v^4 + \tfrac{7}{3} v^6 + \tfrac{19}{4} v^8 + \tfrac{61}{5} v^{10} \right\}$$

Note: Used $\cosh^2(\beta J) = \frac{1}{1 - \tanh^2 \beta J} = \frac{1}{1 - v^2}$ and $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$

Note: F is extensive because only connected diagrams contribute to F