High temperature series for Ising chain.

Recall our high temperature series expansion for the Ising chain,

$$Z = 2^N \cosh^N(\beta J) \cosh^N(\beta h) \left(1 + N u^2 v + \ldots\right)$$

with $u = \tanh(\beta h)$ and $v = \tanh(\beta J)$.

1. Notice the free energy can be written as $F = F_0 + \Delta F$, where $F_0 = -kT \log\left[2^N \cosh^N(\beta J)\cosh^N(\beta h)\right]$ and $\Delta F = -kT \log\left[1 + N u^2 v\right]$. Expand ΔF up to first order in v. Is the free energy extensive (i.e. proportional to N)?

2. Use this result to calculate the magnetization per spin, $M(J, h)$ accurately up to first order in h.

3. Now calculate the magnetic susceptibility χ. Compare with the exact susceptibility obtained from the transfer matrix.

1. $F_0 = -NkT \ln\left[2 \cosh(\beta J) \cosh(\beta h)\right]$ is extensive.

As written $\Delta F = -kT \ln\left[1 + N u^2 v\right]$ is not extensive, but up to 1st order in v, $\Delta F = -NkT u^2 v$ is extensive.

2. Expand F to order h^2:

$u \equiv \tanh(\beta h) = \beta h + O(h^2) \Rightarrow \Delta F = -\frac{N}{kT} v h^2$

$\cosh(\beta h) = 1 + \frac{1}{2}(\beta h)^2 \Rightarrow F_0 = -NkT \ln(2 \cosh \beta J) - \frac{N}{2kT} h^2$

$M = -\frac{1}{N}\frac{\partial F}{\partial h} = \frac{1 + 2v}{kT} h$

3. High T: $\chi = \frac{1 + 2\tanh(\beta J)}{k_B T} = \frac{1}{kT}\left(1 + 2\beta J - \frac{2}{3}(\beta J)^3 + \cdots\right)$

Transfer matrix: $\chi = \frac{e^{2\beta J}}{kT} = \frac{1}{kT}\left(1 + 2\beta J + \frac{1}{2}(2\beta J)^2 + \cdots\right)$