2D Square Lattice Ising Self Duality

Low T: $$Z = \Omega_0 e^{2N\beta J} \underbrace{\left\{1 + N x^4 + 2N x^6 + \tfrac{1}{2} N(N+9) x^8 + \cdots \right\}}_{\sum (e^{-2\beta J}) = \sum_{\text{islands of flipped spins}} (e^{-2\beta J})^{(\text{perimeter of island})}}$$

$x = e^{-2\beta J}$

High T: $$Z = 2^N \cosh^{2N}(\beta J) \underbrace{\left\{1 + N v^4 + 2N v^6 + \tfrac{1}{2} N(N+9) v^8 + \cdots \right\}}_{\sum (\tanh(\beta J)) = \sum_{\text{graphs with even \# bonds on each site}} \tanh(\beta J)^{(\text{size of graph})}}$$

$v = \tanh(\beta J)$

$\beta J \to J$ for simplicity: low T → high J, high T → low J

Exact free energy:

$$-F = \frac{1}{N} \ln Z = \ln(\Omega_0 e^{2J}) + \frac{1}{N} \ln \sum (e^{-2J})$$

$$-F = \frac{1}{N} \ln Z = \ln(2\cosh \beta J) + \frac{1}{N} \ln \sum (\tanh J)$$

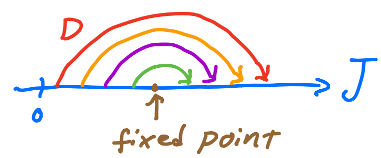
same Σ! $e^{-2J} \leftrightarrow \tanh J$?

Duality Mapping:

D: low J (high T) $\longrightarrow$ high J' (low T')

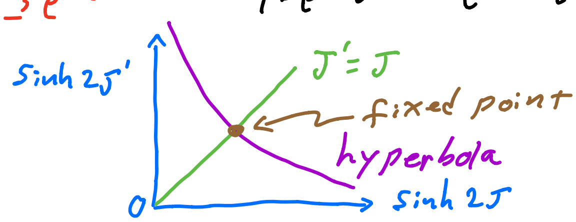
D: $\tanh J \longrightarrow e^{-2J'}$

$$J' = -\frac{1}{2} \ln\{\tanh(J)\}$$



Note: $$\sinh(2J) = 2\sinh(J)\cosh(J) = 2\tanh(J)\cosh^2(J) = \frac{2\tanh(J)}{1-\tanh^2(J)}$$

substitute $\tanh J \to e^{-2J'}$ $\Rightarrow$ $$= \frac{2e^{-2J'}}{1-e^{-4J'}} = \frac{2}{e^{2J'}-e^{-2J'}} = \frac{1}{\sinh(2J')}$$



Suppose F is singular at $J_c \Rightarrow$ also singular at J_c'

Fixed point of D: $D(J_c) = J_c$ $\quad e^{-2J_c} = \tanh J_c = \dfrac{1-e^{-2J_c}}{1+e^{-2J_c}}$

quadratic equation for e^{-2J_c}

Solution: $e^{-2J_c} = \sqrt{2} - 1$

$= 1/(\sqrt{2}+1) = 0.414 \equiv x_c$

$J_c = \frac{1}{2} \ln(\sqrt{2}+1) = 0.441$

$k_B T_c = 2.269\ J \quad (2.269 = 1/0.441)$