Low T energy and heat capacity coefficients

Low T partition function $Z_L = 2e^{2NJ/T} \sum(x)$, $x = e^{-2J/T}$

Free energy per spin: $-\beta f_L = \lim_{N\to\infty} \frac{1}{N} \ln Z_L$

Coefficients f_n

$$= \frac{1}{N}\ln 2 + \frac{2J}{T} + 1 \cdot x^4 + 2x^6 + \frac{9}{2}x^8 + 12\,x^{10}$$

note only $O(N)$ terms
N^2, etc. cancel in $\ln\sum$

Internal energy per spin:

$$u = \frac{\partial}{\partial\beta}(\beta f) = -2J - \frac{\partial}{\partial\beta}\sum_n f_n x^n$$

Note: $\frac{\partial}{\partial\beta} = \frac{\partial x}{\partial\beta}\frac{\partial}{\partial x} = (-2J)\,x\frac{\partial}{\partial x}$ $\Rightarrow$ $u = -2J + 2Jx\sum_n n f_n x^{n-1}$

$$= -2J + 2J\sum_n \underbrace{n f_n}_{u_n \equiv n f_n} x^n$$

$$= -2J\,[1 - 4x^4 - 12\,x^6 - 36\,x^8 - \cdots]$$

Specific heat:

$$c = \frac{\partial u}{\partial T} = -\frac{1}{T^2}\frac{\partial u}{\partial\beta} = \frac{2Jx}{T^2}\frac{\partial u}{\partial x} = \left(\frac{2J}{T}\right)^2\left(x\frac{\partial}{\partial x}\right)^2(-\beta f) = \left(\frac{2J}{T}\right)^2\sum_n \underbrace{n^2 f_n}_{c_n = n^2 f_n} x^n$$

Relation of Low T coefficients f_n to High T coefficients:

High T $Z_H = 2^N \underbrace{\cosh^{2N}(\beta J)}_{(1-v^2)^{-N}} \sum(\underbrace{\tanh(\beta J)}_{v})$

$$-\beta f_H = \ln 2 - \underbrace{\ln(1-v^2)}_{\sum_{m=1}^{\infty}\frac{1}{m}v^{2m}} + \underbrace{\ln\sum(v)}_{\sum_n f_n v^n} = \ln 2 + \sum_{m=1}^{\infty}\left(\frac{1}{m} + f_{2m}\right)v^{2m}$$