NAME: SOLUTIONS

33-342 Thermal Physics II   Final Exam   Friday, May 8, 2020

0. Have you completed your FCE, or will you do so? Yes (1 point)/No (0 points)

1. Consider the Ising model on a hypercubic lattice of dimension $d > 1$ (*i.e.* square, simple cubic, *etc.*.), with Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j.$$

Let $M = \sum_i \sigma_i$ be the magnetization of a configuration $\vec{\sigma} = \{\sigma_i\}$, and let E be the value of the Hamiltonian H for this configuration.

(a) Express $\partial\langle M\rangle/\partial T$ in terms of averages involving M and E.

Answer: Start with

$$\langle M \rangle = \frac{1}{Z} \sum_{\vec{\sigma}} M e^{-\beta H}$$

Recall that $\partial/\partial\beta = -k_{\mathrm{B}}T^2 \partial/\partial T$, and that $\langle E \rangle = -\partial Z/\partial \beta$. Thus we find

$$\frac{\partial \langle M \rangle}{\partial T} = -\frac{1}{k_{\mathrm{B}}T^2} \left(\langle EM \rangle - \langle E \rangle \langle M \rangle \right).$$

(b) According to a low temperature series expansion (*e.g.* see solutions to the 2018 final exam), the magnetization $M \approx N - 2Ne^{-4d\beta J}$. Briefly explain the factor of $2N$ multiplying the exponential, and the factor of $4d$ in the exponent.

Answer: Flipping a single spin reduces M replaces +1 with -1 for a net change of 2. There are N individual spins that can flip, yielding a factor of $2N$ multiplying the flip probability. A single spin flip breaks $2d$ bonds, for a net energy change of $\Delta E = 4dJ$. Thus $\exp -\beta\Delta E = \exp(-4d\beta J)$ is the flip probability (at low temperature).

(c) With what critical exponent do you expect $\partial M/\partial T$ to vary as $T \to T_c^-$? Does $\partial M/\partial T$ vanish, or diverge?

Answer: Since $M \sim (T_c - T)^\beta$, we have $\partial M/\partial T \sim (T_c - T)^{\beta - 1}$. We know that $\beta \le 1/2$ in all dimensions, so $\partial M/\partial T$ must diverge.

2. The following questions study spins interacting with effective fields due to their environments.

(a) An Ising spin $\sigma = \pm 1$ is surrounded by spins that are fixed in the up ($+1$) orientation. It interacts with each of its z neighbors with a bond energy $-J\sigma$ with $J > 0$. What is the probability that the spin takes value σ?

Answer:

$$P(\sigma) = \frac{e^{\beta J z \sigma}}{e^{\beta J z} + e^{-\beta J z}} = \frac{e^{\beta J z \sigma}}{2\cosh \beta J z}$$

(b) Determine the low and high temperature limits of $\langle \sigma \rangle$, in each case keep the limiting value and the first temperature-dependent correction.

Answer: Using (a) to take the average yields

$$\langle \sigma \rangle = \frac{e^{\beta J z} - e^{-\beta J z}}{e^{\beta J z} + e^{-\beta J z}} = \tanh(\beta J z)$$

At low temperature, write $\tanh(x) = (1 - e^{-2x})/(1 + e^{-2x}) \approx 1 - 2e^{-2x}$, so that $\langle \sigma \rangle = 1 - 2e^{-2\beta J z}$, same as in part 1(b).

(c) The mean field self-consistent equation for magnetization is $m = \tanh(\beta J z m)$. Evaluate the limiting value and first temperature-dependent correction for m as $T \to 0$. Be sure to justify any assertions or approximations that you make.

Answer: Taking the same expansion for tanh of large arguments we obtain the implicit transcendental equation

$$m = 1 - 2e^{-2\beta J z m}.$$

Since $m \to 1$ as $T \to 0$, we may substitute $m = 1$ in the exponential to obtain $m \approx 1 - e^{-2\beta J z}$.

(d) In the presence of an external magnetic field, the self-consistent equation is $m = \tanh(\beta(Jzm + h))$. Calculate the two leading terms in m as $T \to \infty$ at fixed $h > 0$. Again, be sure to justify any approximations or assertions that you make.

Answer: For small arguments we expand $\tanh x \approx x - x^3/3$ and thus obtain the self-consistent cubic equation

$$m = \beta(Jzm + h) - \frac{1}{3}(\beta(Jzm + h))^3.$$

Since $\beta \to 0$, we tentatively drop the cubic term and solve for

$$m = \frac{\beta h}{1 - \beta J z} \approx (\beta + \beta^2 J z) h$$

Since $\beta^3 \ll \beta^2$, we were justified in dropping the cubic term.

3. For the second year in a row, the universe has collapsed and you are stuck in a space of dimension d with volume $V = L^d$ at a high temperature T. The photon dispersion relation became nonlinear again, so that $\omega = ck^s$, with c and s positive constants. The photons now obey a conservation law so that their total number N is fixed, and their chemical potential μ need not vanish. Luckily, no other physical laws have changed. For example, the occupation of a bosonic state of energy $E = \hbar\omega$ is still given by

$$\langle n_\omega \rangle = \frac{1}{e^{\beta(\hbar\omega - \mu)} - 1}.$$

To survive you must extract energy from excited state photons.

(a) Demonstrate that the photon density of states is $D(\omega) = A\omega^{d/s-1}$, and determine the value of A.

Answer: The allowed wavevectors are $\mathbf{k} = \frac{\pi}{L}\mathbf{n}$, with $\mathbf{n}$ a d-dimensional vector of positive integers. The number of states with frequency less than ω is

$$\mathcal{N}(\omega) = \frac{1}{2^d}\left(\frac{L}{\pi}\right)^d V_d \left(\frac{\omega}{c}\right)^{d/s}$$

where the $1/2^d$ factor comes from the restriction to positive integers, and V_d is the prefactor for d-dimensional volumes (*i.e.* $V_1 = 2$, $V_2 = \pi$, and $V_3 = 4\pi/3$). The density of states is

$$D(\omega) = \frac{d\mathcal{N}}{d\omega} = \frac{d}{s}\frac{\mathcal{N}(\omega)}{\omega} = A\omega^{d/s-1}, \quad A = \frac{1}{2^d}\left(\frac{L}{\pi}\right)^2 \left(\frac{d}{sc^s}\right) V_d$$

assuming just a single polarization state. Multiply by an appropriate factor if your strange photons have multiple polarization states.

(b) Write down (but do not attempt to evaluate) an integral relationship between the number of photons present, N, and their chemical potential, μ.

Answer: The total number is the integral of the number at each frequency

$$N = \int \mathrm{d}\omega \frac{D(\omega)}{e^{\beta(\hbar\omega - \mu)} - 1}$$

(c) Unfortunately, after you completed the previous calculation, the temperature dropped and some of the photons have Bose condensed! How many excited state photons, N_e, remain out of the original N that were present? You may wish to look in Wikipedia to find the integral representation of the Riemann zeta function as an aid to evaluate the number.

Answer: Bose condensation sends $\mu \to 0^-$. The excited state photons are still counted in the value of the integral, hence

$$N_e = \int \mathrm{d}\omega \frac{D(\omega)}{e^{\beta\hbar\omega} - 1} = A\zeta(d/s)$$

(d) The occurrence of Bose condensation taught you some useful information concerning d and s. What did you learn?

Answer The zeta function $\zeta(p)$ diverges at $p = 1$ and is negative for $p < 1$. Hence we must have $d > s$. This can also be checked by expanding $e^{\beta\hbar\omega} - 1$ for small ω and the looking for the condition for convergence of the integral at small ω.

(e) Imagine that you have a d-dimensional solid body of N atoms in volume $V = L^d$. The atoms interact with a strange potential leading to the dispersion relation $\omega = ck^s$.

(i) Write down a formal expression that determines the Debye frequency ω_D.

Answer: (i) The Debye frequency is set by the requirement that $\mathcal{N}(\omega_D) = N$, assuming one degree of freedom per atom.

(ii) The Debye frequency v aries with density as $(N/V)^q$ for some power q. Determine q and check that it has the expected value for $d = 3$ and $s = 1$ ($i.e.$ ordinary phonons).

Answer: (ii) Since $\mathcal{N}(\omega) \sim V\omega^{d/s}$, we have $\omega_d \sim (N/V)^{s/d}$. For ordinary phonons $s/d = 1/3$.