Exact calculation of finite Ising chain

1. The Hamiltonian for the Ising chain in 1D with open boundary conditions is

$$H = -J \sum_{j=1}^{N-1} \sigma_j \sigma_{j+1} - h \sum_{j=1}^{N} \sigma_j, \quad \sigma_j = \pm 1$$

   Write down formal expressions (*e.g.* expressed as summation but not evaluated) for the partition function Z, the free energy F, the internal energy U, the heat capacity C, the magnetization $M = \sum_j \sigma_j$, and the magnetic susceptibility $\chi = \partial M / \partial h$.

2. For a given chain length N, how many terms are in these sums?

3. Devise a scheme for sequentially listing all possible states of the chain $\vec{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ that can be programmed in Python.

4. Write a program to calculate U, C and χ as functions of temperature for an arbitrary value of N. Run the program for N=5 and 10. How does the run time vary with N? For each N, plot your results for $k_B T / J$ ranging from 0.1 to 2. Compare the behavior for h/J=0, 0.1, 0.2. For h=0 compare U and C with the exact analytic result.