Correlation functions of the Ising chain

This problem considers the Ising chain with Hamiltonian

\[ H = -J \sum_{j=1}^{N} \sigma_j \sigma_{j+1} - \sum_{j=1}^{N} h_j \sigma_j \]

Note that the magnetic field may differ among sites, although we will mainly be interested in the case where all fields vanish, \( \vec{h} = \vec{0} \). Assume periodic boundary conditions.

(a) Show that \( m_i \equiv \langle \sigma_i \rangle = -\partial F / \partial h_i \)

(b) Show that \( \chi_{ik} \equiv \partial m_i / \partial h_k = -\partial^2 F / \partial h_i \partial h_k = \beta (\langle \sigma_i \sigma_k \rangle - \langle \sigma_i \rangle \langle \sigma_k \rangle) \).

(c) Show that \( \chi_{ik} \) is translation invariant when \( \vec{h} = \vec{0} \). Hence \( \chi_{ik} = g(x) \), where \( x = k - i \). Interpret the meaning of \( g(x) \).

(d) Show that

\[ \sum_{x=-N/2}^{N/2} g(x) = \chi_0, \]

with \( \chi_0 \) the zero field magnetic susceptibility.