Quantum Gas Problem 3: Heat capacity near the Bose-Einstein transition

The next quantity that we wish to compute and plot is the heat capacity at constant N and V, that is

$$C_{V,N} = T \left(\frac{\partial S}{\partial T} \right)_{V,N} = \left(\frac{\partial U}{\partial T} \right)_{V,N} , \tag{1}$$

Keeping V constant is not a problem, we will ignore the volume dependence to simplify the notation, writing C_N when we really mean $C_{V,N}$.

The difficulty is that the natural quantity to calculate is the natural quantity to calculate is

$$\left(\frac{\partial U}{\partial T} \right)_{\mu} . \tag{2}$$

QUESTIONS:

1. Prove the following thermodynamic identity.

$$C_N = \left(\frac{\partial U}{\partial T} \right)_{\mu} - \left(\frac{\partial U}{\partial \mu} \right)_{T} \left(\frac{\partial N}{\partial T} \right)_{\mu} \Big/ \left(\frac{\partial N}{\partial \mu} \right)_{T} \tag{3}$$

2. Find explicit expressions for all quantities on the right side of Eq. (3) in terms of sums over $\vec{n}$.

3. Write a computer program to plot $C_N(T)$ from $T = 0$ up to $2T_{BE}$. Run the program for system sizes from $N = 1 \ldots 4$ at fixed density $N/V = 1$. Include, for comparison, the predicted heat capacity below T_{BE}, $C = 1.925N(T/T_{BE})^{3/2}$.