Example Problem (S'17 E2)

Simplified model with B-E transition: free particles in 2D with band gap.

Note: DOS \( D(E) = X \cdot V \) indy of \( E \) in 2D for \( E < E_0 \)

\[ D(E) = X \cdot V \cdot \left( e^{\frac{E-E_0}{k_B T}} - 1 \right)^{-1} \]

\[ N = N_0 + N^* \text{ doping } \mu \]

\[ N^* = \int_{E_0}^{E} \frac{dE}{D(E)} \left( e^{\frac{E-E_0}{k_B T}} - 1 \right) \]

(a) What equation determines chemical potential?

\[ \mu = \langle n_e \rangle = \left( e^{\frac{E-E_0}{k_B T}} - 1 \right)^{-1} \]

\[ \mu = \frac{1}{N_0} \int_{E_0}^{\infty} \frac{dE}{D(E)} \left( e^{\frac{E-E_0}{k_B T}} - 1 \right) \]

(b) Allowable range of \( \mu \)? \( \mu \to -\infty \) for sufficiently small \( N \) and large \( V \)

\[ N^* \to 0 \quad \text{and} \quad N_0 \to 0 \quad \Rightarrow \quad \mu < E_0 < 0 \]

(e) Determine \( T_{BE} \)

\[ A \cdot T_{BE}, \quad N \ll N^* \Rightarrow N^* \ll N \]

\[ N = X \cdot V \int_{E_0}^{\infty} \frac{de}{e^{E/k_B T} - 1} = \mu = e^{E_0/k_B T} \]

\[ \mu = e^{E_0/k_B T} \]

\[ \frac{\beta e N}{X \cdot V} = \ln \left( 1 - e^{-e_0/e} \right) \]

\[ C = 1 - e^{-e_0/e} \]

(Transverse equation for \( \mu \))

(1)

\[ \text{Example: if } N = e_0 \cdot X \cdot V \quad \text{then} \quad \mu = e^{E_0/k_B T} \quad \Rightarrow \quad k_B T_e = E_0 \ln 2 \]

\[ \text{Note: if } E \to 0 \quad \text{then} \quad T_{BE} \to 0 \quad \text{for band edges in outer electrons of 2D or 1D system} \]

(1) Find \( N_0(T) \) for \( T < T_{BE} \)

\[ N_0 = N - N^* = N \left( 1 + \ln \left( 1 - e^{-e_0/e} \right) \right) = \frac{1}{N} \ln \frac{e_0}{1 - \frac{e_0}{N}} \]

\[ N_0 = \frac{e_0}{1 - \frac{e_0}{N}} \]

(2) Find \( \mu(T) \) as \( T \to 0 \)

\[ N_0 = N \left( 1 - \frac{X \cdot V}{e^{E_0/k_B T}} \right) \]

\[ \mu = \ln \left( 1 - \frac{X \cdot V}{e^{E_0/k_B T}} \right) \]

\[ \mu = e^{E_0/k_B T} - \ln \left( 1 - \frac{X \cdot V}{e^{E_0/k_B T}} \right) \]
Bose-Einstein condensation temperatures

Bose-Einstein and superfluid transitions have been observed in many materials. Evaluate the transition temperature \( T_{BE} = (2\pi \hbar^2 / k_B m)(N/2.612V)^{1/2} \) for the gases listed below. In each case, calculate the mean interatomic spacing \( R = (V/N)^{1/3} \) and compare with the de Broglie quantum wavelength \( \lambda = \sqrt{2\pi \hbar^2 / m k_B T} \) at \( T = T_{BE} \). Also compare your calculated \( T_{BE} \) with the experimental transition temperature and discuss.

1. Bose condensation was reported in 1998 by Kleppner and Greytak (MIT) at \( T=50 \mu K \) in a gas of \(^1\)H with density \( n = N/V = 1.8 \times 10^{14} \text{cm}^3 \). Compare \( R \) and \( \lambda \) with the Bohr radius of \( a = 0.529 \text{Å} \).

2. Kapitza discovered superfluidity in liquid \(^4\)He at \( T=2.17 \text{K} \) in 1937. The atomic volume \( v = V/N = 46 \text{Å}^3 \).

3. Bose condensation in alkali metal atoms was discovered independently at 170 nK in \(^{87}\)Rb in June 1995 by Wieman and Cornell (at JILA, Colorado) and four months later at 2 nK in \(^{23}\)Na by Ketterle (MIT). Estimated densities were \( 1.5 \times 10^{14} / \text{cm}^3 \) for \(^{23}\)Na and \( 2.5 \times 10^{12} / \text{cm}^3 \) for \(^{87}\)Rb.

4. Except for \(^4\)He, all the above gases come from the first column of the periodic table.

Why are these gases popular targets for study?

1. \(^1\)H \( m = 1 \text{AMU} \) \( T_{BE} = 51 \mu K \) \( R = \frac{3 \lambda}{\lambda(0)} = 1.7 \times 10^{-7} \text{m} \) = 2100 Bohr \( \gg a \) = 1 Bohr

\( \lambda(T_{BE}) = 2.4 \times 10^{-10} \text{m} \)

\( \Delta^2 / R^2 = n \lambda^3 = 2.6 \left< \right> \)

\( \lambda(T_{BE}) = 5 \lambda \text{ is an identity for any Bose gas} \)

2. \(^4\)He \( m = 4 \text{AMU} \) \( T_{BE} = 3.14 \text{K} \) \( R = 3.6 \times 10^{-10} \text{m} \)

\( \lambda = 5 \text{Å} \) \( \text{diam(He)} = 3 \text{Å} \)

3. \(^{13}\)Na \( m = 23 \text{AMU} \) \( T_{BE} = 2.6 \mu K \) \( R = 1.9 \times 10^{-7} \text{m} \) = 1900 Å

\( \lambda = 2700 \text{Å} \) \( \text{diam(Na)} = 20 \text{Å} \)

4. \(^{87}\)Rb \( m = 87 \text{AMU} \) \( T_{BE} = 3.4 \text{nK} \) \( R = 7400 \text{Å} \)

\( \lambda = 4500 \text{Å} \) \( \text{diam(Rb)} = 27 \text{Å} \)

\( ^1\)st exp.

Note difficulty in priming \( n \) in experiment

4. Alkali metals and hydrogen all show simple volume electron \( = 5 \text{ simple optical spectra, easy to cool and measure} \)

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