

NAME: _____ SOLUTIONS

33-448 Solid State Physics

Midterm #3

Monday, April 4, 2016

Note that problems 1 and 2 require brief written explanations of reasoning in a few sentences without lengthy derivation.

1. Heat capacity

The electronic density of states varies as $D(E) \sim E^{-1+d/2}$ in spatial dimension d , yet the electronic heat capacity varies as $C_{el} \sim T^p$, where the power p is independent of dimension d . Briefly explain the value of p . Why does it not depend on dimension, while the power law for vibrational heat capacity (C_{vib}) does depend on d ?

Answer: Electrons absorb heat by excitation from kT below the Fermi energy to kT above. The number of excitable electrons is thus $D(E_F)kT$, so that $E_{el} \sim D(E_F)(kT)^2$ and hence $C_{el} \sim D(E_F)k^2T$. The Fermi energy E_F is independent of temperature owing to Pauli exclusion. In contrast, vibrations absorb heat by creating phonons of energy $\hbar\omega$ with frequency up to kT/\hbar , and the phonon density of states vanishes at low frequency.

2. Pauli paramagnetism

Every electron has magnetic moment $\pm\mu\hat{z}$. In the presence of a magnetic field $\mathbf{B} = B\hat{z}$, electrons with moment “up” (parallel to the magnetic field) drop in energy by $-\mu B$, while those with moment down (opposite to the field) rise in energy by $+\mu B$. Some electrons flip their moments to align with the applied field so as to reduce the total energy. This effect is illustrated in the figure below, taken from Kittel chapter 11. Let $T = 0\text{K}$ (absolute zero) in the following.

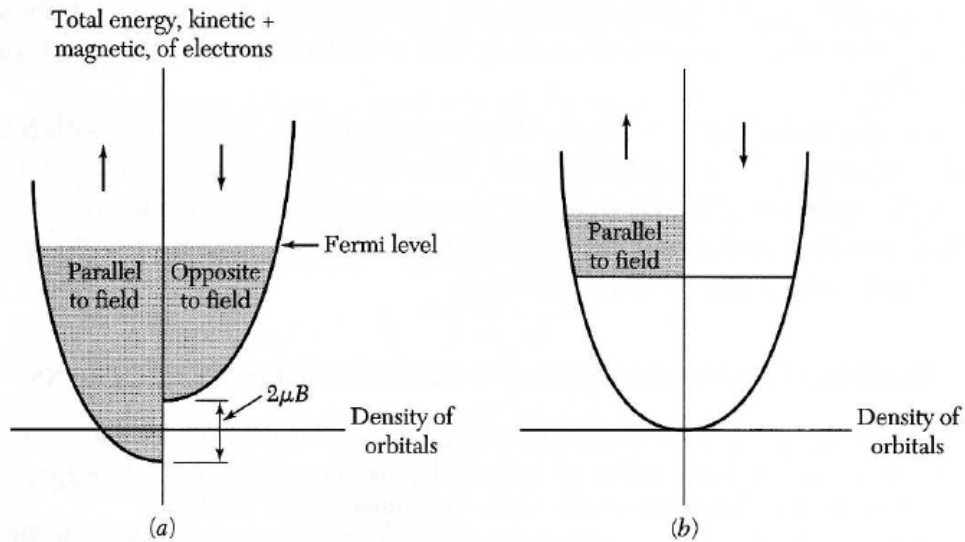


Figure 10 Pauli paramagnetism at absolute zero; the orbitals in the shaded regions in (a) are occupied. The numbers of electrons in the “up” and “down” band will adjust to make the energies equal at the Fermi level. The chemical potential (Fermi level) of the moment up electrons is equal to that of the moment down electrons. In (b) we show the excess of moment up electrons in the magnetic field.

Let N_+ be the number of spin up electrons, N_- the number of spin down, and $M \equiv (N_+ - N_-)\mu$ be the net magnetization. For weak applied fields, the magnetization $M = [\mu]^p [D(E_F)]^q [B]^r$, where p, q and r are powers of moment, Fermi level density of states and field, respectively. Without any explicit derivation, state the values of p, q and r and briefly justify your assertions.

Answer:

$p = 2$: because one power of μ comes from the shift in energy (and thus the effective Fermi levels for moments up and down), while the other is the factor of μ in the definition of M . Alternatively, reversing the moment of each electron prior to applying the field yields a system physically equivalent to the starting case, and hence its magnetic response must be unaltered.

$q = 1$: because the electrons whose moments reverse must lie close to the Fermi energy, and their number is proportional to $D(E_F)$.

$r = 1$: because the energy shift (and hence the number of flipped electrons) is proportional to B . Alternatively, reversing the applied field must reverse the magnetization. This is actually the *definition* of paramagnetism.

3. Frequency-dependent conductivity (adapted from Kittel #6.6)

Recall the drift velocity equation $m(d/dt + 1/\tau)v = qE$ for particles of mass m and charge q . Let the electric field oscillate as $E = \mathcal{E}_0 e^{-i\omega t}$ with $\omega > 0$ a fixed value.

(a) Derive a formula for the frequency-dependent complex conductivity $\sigma(\omega) \equiv j/E$, where j is the current density. Show the steps of your derivation, define any additional quantities you may need to introduce, and express your answer in the form $\sigma(\omega) = \sigma_0 f(\omega)$, where σ_0 is the static conductivity.

Answer: Assume $v = v_0 e^{-i\omega t}$ and solve for $v_0 = \mathcal{E}_0 / (-i\omega + 1/\tau)$. Writing $j = qnv$ and placing the complex factor in the numerator yields

$$j = \left(\frac{nq^2\tau}{m} \right) \left(\frac{1 + i\omega\tau}{1 + (\omega\tau)^2} \right)$$

(b) Show that the current j falls out of phase with the electric field E in the limit of large relaxation time τ . What causes this phase lag, and what impact does it have on the average energy dissipation?

Answer: For large τ the current approaches $j = inq^2/m\omega$. The factor of $i = \sqrt{-1}$ causes a 90° phase lag relative to the electric field. The reason is that the charges *accelerate* when the field is large, but they reach their maximum velocity as the field vanishes. Since the product of field (force) times current (velocity) is equally often positive and negative, energy does not dissipate on average. This is consistent with the long relaxation time which implies diverging real conductivity (vanishing resistance).