NAME: SOLUTIONS

33-448 Solid State Physics   Midterm #1   Wednesday, Feb. 20, 2017

Please read each question carefully before answering. Do not do any unnecessary work - it will waste time and not gain you points. Be sure to attempt every part, as most can be solved independently.

1. Chemical potential of 2D electrons (adapted from Simon, #4.5)

(a) Derive the density of states $D(E)$ of electrons in two dimensions.

Answer: The k-space density of wavevectors is $(L/2\pi)^2$. Hence the number of states below energy E is (including a factor of 2 for spin)

$$\begin{aligned}\mathcal{N}(E) &= 2\left(\frac{L}{2\pi}\right)^2 \pi \left(\sqrt{2mE/\hbar^2}\right)^2 \\ &= \frac{L^2 mE}{\pi\hbar^2}.\end{aligned}$$

The density of states is obtained by differentiating,

$$D(E) = \mathrm{d}\mathcal{N}/\mathrm{d}E = \frac{L^2 m}{\pi\hbar^2},$$

leading to a result that is independent of E.

(b) Write down an identity that relates the chemical potential μ to the 2D density of electrons n. Your identity may contain derivatives, integrals or similar complicated features that you do not need to evaluate. All quantities utilized must be defined.

Answer: We can count the total number of electrons N by integrating the density of occupied states,

$$N = \int_0^\infty \mathrm{d}E\ D(E)\ f_T(E),$$

where

$$f_T(E) = 1/\left(\exp\left((E-\mu)/k_B T\right) + 1\right)$$

is the Fermi-Dirac occupation function. Substituting our result from part (a), and defining $n = N/L^2$,

$$n = \frac{m}{\pi\hbar^2} \int_0^\infty \frac{\mathrm{d}E}{\exp\left((E-\mu)/k_B T\right) + 1}.$$

If we wished we could easily evaluate the integral, yielding

$$n = \frac{m k_B T}{\pi \hbar^2} \log\left(1 + \exp\left(\mu/k_B T\right)\right)$$

and hence an explicit formula for chemical potential

$$\mu(T) = k_B T \log \exp\left(n\pi\hbar^2/m k_B T\right) - 1,$$

from which one can show that $mu(T)$ is nearly independent of T provided $\mu/k_B T \ll 1$.

(c) Evaluate your expression from part (b) in the limit $T \to 0$ to obtain an expression for $\mu(T=0)$.

Answer: Because the Fermi-Dirac occupation functionb ecomes a step function as $T \to 0$, we write

$$N = \int_0^\mu \mathrm{d}E\ D(E) = \frac{\mu L^2 m}{\pi\hbar^2},$$

from which we obtain $\mu(T=0) = n\pi\hbar^2/m$.

We could have obtained $\mu(T=0)$ without going through part (b) is *via* the Fermi energy $E_F = \lim_{T\to 0} \mu(T)$. Then we obtain E_F from part (a) by setting $N = \mathcal{N}(E_F)$, immediately yielding $E_F = n\pi\hbar^2/m$.

2. Problems involving electrons and impurities

(a) (Matthiessen's rule)

Suppose a metal scatters electrons by two different mechanisms: impurities scatter electrons with mean free time τ_i; electrons scatter each other with mean free time τ_e. Derive an expression for the conductivity. Note: the solution is very short but slightly tricky.

> **Answer:** In a time interval t large compared to τ_i and τ_e the electron will have a total of $n = t/\tau_i + t/\tau_e$ collosions. Hence the mean free time τ is
>
> $$\frac{1}{\tau} = \frac{1}{\tau_i} + \frac{1}{\tau_e}.$$
>
> We then apply the usual Drude result
>
> $$\sigma = \frac{ne^2\tau}{m}.$$
>
> Alternatively, we could have written $\sigma = 1/\rho$, where we add the resistivities $\rho_i + \rho_e$ in series.

(b) (Electron-impurity collisions)

Consider N_e electrons traveling at speed s in a material that contains N_i randomly placed impurities in a volume V. The impurities are atoms of radius R_i, and the electrons can be treated as point particles. Derive an expression for the electron mean free time due to impurity scattering, τ_i. Hint: it might be easier to think of the electrons as stationary and the impurities as moving!

> **Answer:** Consider a frame of reference with N_e randomly placed electrons at rest and N_i impurity atoms moving at speed s. In a time interval T, each impurity will collide with all electrons contained in a cylinder of radius R_i and length sT. The number of electrons in each cylinder is $N = (\pi R_i^2)(sT)(N_e/V)$, yielding a total number of collisions $C = N_i N$, for a number of collisions per electron of $C/N_e = n_i \pi R_i^2 sT$, with $n_i = N_i/V$ the density of impurities. The mean free time is $\tau_i = T/(C/N_e) = 1/(n_i s \pi R_i^2)$.