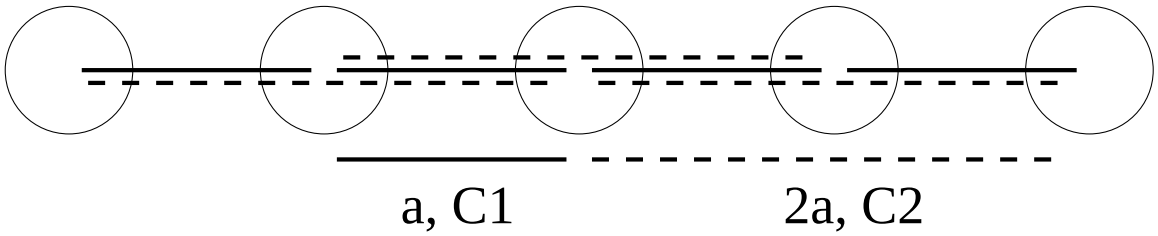
NAME: SOLUTIONS

33-448 Solid State Physics Midterm #2 Monday, March 27, 2017

Please read each question carefully before answering. Do not do unnecessary work - it will waste time and might cost you points. Be sure to attempt every part, as some can be solved independently of others.

1. Dispersion relation for next-nearest neighbor chain (adapted from Simon #9.3)

A monatomic chain of atoms with mass m has nearest neighbor springs of unstretched length a and spring constant C_1, and second neighbor springs of unstretched length $2a$ and spring constant C_2.



(a) Write down the equation of motion for the n^{th} atom in the chain.

Answer:

$$m\ddot{\delta x}_n = C1\left(\delta x_{n+1} + \delta x_{n-1} - 2\delta x_n\right) + C2\left(\delta x_{n+2} + \delta x_{n-2} - 2\delta x_n\right)$$

(b) Determine the dispersion relation $\omega(k)$. Hint: the solution is almost trivial if you remember that the functional form of normal modes is governed by the symmetries of the system under consideration.

Answer: From the translational symmetry of the chain we know the solutions take the form $\delta x_n = \exp\left(i(kna - \omega t)\right)$. Plugging into the equation of motion yields

$$-m\omega^2 = 2C_1\left(\cos\left(ka\right) - 1\right) + 2C_2\left(\cos\left(2ka\right) - 1\right)$$

(c)

i. What is the sound speed v_s?

Answer: Sound refers to vibrations with $k \ll \pi/a$. In this case we may expand

$$\cos ka \approx 1 - \frac{1}{2}(ka)^2, \quad \cos 2ka \approx 1 - \frac{1}{2}(2ka)^2$$

leading to

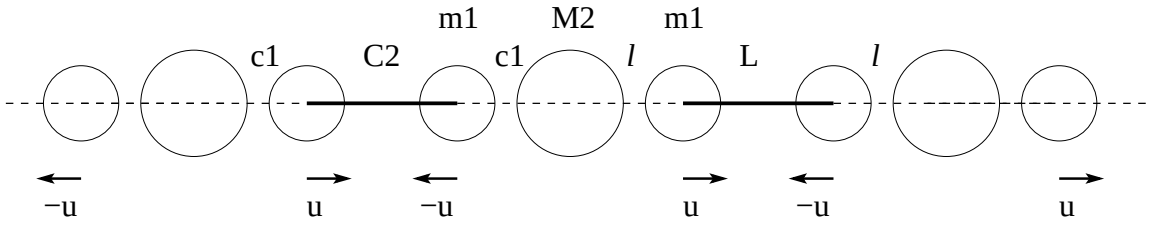
$$\omega \approx \sqrt{(C_1 + 4C_2)/m}\; ka, \quad v_s = \sqrt{(C_1 + 4C_2)/m}\; a.$$

ii. Discuss the relative proportions of C_1 and C_2 in your expression for v_s.

Answer: The value of C_2 contributes four times as much as the value of C_1. This is because the second neighbor bond length changes twice as much as the near neighbor bond under uniform stretch or compression, and thus its energy grows four times faster. Note that stretch and compression become uniform in the $k \to 0$ limit corresponding to sound.

2. Triatomic chain

(a) Consider the triatomic chain illustrated below. The masses are m_1 and $M_2 > m_1$, the bond lengths are ℓ and $L > \ell$, and their spring constants are c_1 and $C_2 > c_1$. Sketch the atomic displacements in the highest frequency mode using arrows below all the atoms. Derive (in the simplest way possible!) the frequency of this mode. What is its wavenumber k?



Answer: The small atoms m_1 move in opposition while the central atom M_2 remains fixed. Inserting $u = \exp(-i\omega t)$ into the equation of motion

$$m_1 \ddot{u} = -c_1 u - 2C_2 u$$

yields

$$\omega = \sqrt{(c_1 + 2C_2)/m}.$$

This mode has wavenumber $k = 0$ because every repeat unit has the same motion at the same time.

(b) Use the sound speed $v_s = a\sqrt{C/m}$ of a monatomic chain to write down (without calculation) the sound speed of this triatomic chain. Justify your answer in words, using physical insight instead of equations. Hint: How do spring constants add in series?

Answer: Because the waver number of sound is low (*i.e.* the wavelength is large), the precise atomic structure cannot matter. We can lump all the masses into one and connect the springs into one, yielding an effective mass of $m = 2m_1 + M_2$, and an effective spring constant $1/C = 2/c_1 + 1/C_2$ of length $a = 2\ell + L$. Hence the sound speed is

$$v_s = (2\ell + L)\sqrt{c_1 C_2 / \left((2m_1 + M_2)(2c_1 + C_2)\right)}.$$