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33-658 Quantum I Midterm Exam Oct. 12, 2021

This exam consists of a series of questions concerning three-state quantum systems. Some questions can be answered independently of others. Some questions are quick and easy, while others are more difficult; each is worth 12.5 points.

The three allowed values of $S_{\hat{\mathbf{n}}}$ for a spin-1 particle are $+1, 0, -1$ for any direction $\hat{\mathbf{n}}$ (note that we employ units where $\hbar = 1$). In terms of the eigenstates of S_z, the eigenstates of S_x are

$$
\begin{aligned}
|x^+\rangle &= \frac{1}{2}|z^+\rangle + \frac{1}{\sqrt{2}}|z^0\rangle + \frac{1}{2}|z^-\rangle \\
|x^0\rangle &= \frac{1}{\sqrt{2}}|z^+\rangle - \frac{1}{\sqrt{2}}|z^-\rangle \\
|x^-\rangle &= \frac{1}{2}|z^+\rangle - \frac{1}{\sqrt{2}}|z^0\rangle + \frac{1}{2}|z^-\rangle
\end{aligned}
$$

(a) A beam of spin-1 particles passes through a Z-oriented Stern-Gerlach apparatus. Particles that exit from beam 0 (i.e. those in state $|z^0\rangle$) are sent through an X-oriented Stern-Gerlach. Calculate the exit probabilities in the $|x^+\rangle$, $|x^0\rangle$, and $|x^-\rangle$ channels.

(b) Express the S_x operator for spin-1 as a matrix in the basis of S_z eigenstates, $\{|z^+\rangle, |z^0\rangle, |z^-\rangle\}$ (in that order).

(c) The state $|y^0\rangle$ has the property that neither S_x nor S_z measurements ever yield the value 0. Find the state $|y^0\rangle$.

(d) A magnetic field acts on the spin creating the Hamiltonian

$$H = -\frac{\Omega}{2} S_z.$$

Express the unitary time development operator $U(t)$ as a matrix in the basis of S_z eigenstates.

(e) At time $t = 0$ the spin is in the state $|x^0\rangle$. Determine the probability that the state is $|x^0\rangle$ at time $t > 0$.

(f) The matrix

$$H_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}$$

with $\omega = e^{2\pi i/3}$ is the analog for qutrit states $\{|0\rangle, |1\rangle, |2\rangle\}$ of the Hadamard transformation on qubit states $\{|0\rangle, |1\rangle\}$,

$$H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Show that H_3 is unitary.

(g) (**hard!**) Alice and Bob share a Bell-like state $|B_{00}\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$. Define the matrices acting on the first qutrit of the pair

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and show that the 9 distinct states $|\Psi_{xy}\rangle \equiv (R^x T^y \otimes I)|B_{00}\rangle$ form an orthonormal set, with $0 \leq x, y \leq 2$.

extra space for (g)

(h) Alice chooses a unitary transformation to perform on her half of $|B_{00}\rangle$, then gives her qutrit to Bob, who then performs a measurement on the entangled qutrit pair. Explain (in words) the procedure that Alice and Bob can use so that Alice transmits classical information to Bob. You may assume the claim made in part (g) even if you have not proved it. How much classical information (in units of bits) can Alice transmit *via* operations on her half of the qutrit pair?