NAME: SOLUTIONS

33-658 Quantum I Midterm Exam Oct. 12, 2021

This exam consists of a series of questions concerning three-state quantum systems. Some questions can be answered independently of others. Some questions are quick and easy, while others are more difficult; each is worth 12.5 points.

The three allowed values of $S_{\hat{\mathbf{n}}}$ for a spin-1 particle are $+1, 0, -1$ for any direction $\hat{\mathbf{n}}$ (note that we employ units where $\hbar = 1$). In terms of the eigenstates of S_z, the eigenstates of S_x are

$$
\begin{aligned}
|x^+\rangle &= \frac{1}{2}|z^+\rangle + \frac{1}{\sqrt{2}}|z^0\rangle + \frac{1}{2}|z^-\rangle \\
|x^0\rangle &= \frac{1}{\sqrt{2}}|z^+\rangle - \frac{1}{\sqrt{2}}|z^-\rangle \\
|x^-\rangle &= \frac{1}{2}|z^+\rangle - \frac{1}{\sqrt{2}}|z^0\rangle + \frac{1}{2}|z^-\rangle
\end{aligned}
$$

(a) A beam of spin-1 particles passes through a Z-oriented Stern-Gerlach apparatus. Particles that exit from beam 0 (i.e. those in state $|z^0\rangle$) are sent through an X-oriented Stern-Gerlach. Calculate the exit probabilities in the $|x^+\rangle$, $|x^0\rangle$, and $|x^-\rangle$ channels.

Answer: It is helpful to express the state $|z^0\rangle$ in the X basis. By inspection, $|z^0\rangle = (|x^+\rangle + |x^-\rangle)/\sqrt{2}$. According to the Born rule, the exit probabilities

$$
P(x^+|z^0) = |\langle x^+|z^0\rangle|^2 = \frac{1}{2}, \quad P(x^0|z^0) = |\langle x^0|z^0\rangle|^2 = 0, \quad P(x^-|z^0) = |\langle x^-|z^0\rangle|^2 = \frac{1}{2}.
$$

(b) Express the S_x operator for spin-1 as a matrix in the basis of S_z eigenstates, $\{|z^+\rangle, |z^0\rangle, |z^-\rangle\}$ (in that order).

Answer: This can be achieved by summing up the spectral decomposition. The projectors onto X eigenstates are the dyads

$$
|x^\pm\rangle\langle x^\pm| = \frac{1}{4}\begin{pmatrix} 1 & \pm\sqrt{2} & 1 \\ \pm\sqrt{2} & 2 & \pm\sqrt{2} \\ 1 & \pm\sqrt{2} & 1 \end{pmatrix}, \quad |x^0\rangle\langle x^0| = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},
$$

so that

$$S_x = 1 \cdot |x^+\rangle\langle x^+| \; + \; 0 \cdot |x^0\rangle\langle x^0| - 1 \cdot |x^-\rangle\langle x^-| = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(c) The state $|y^0\rangle$ has the property that neither S_x nor S_z measurements ever yield the value 0. Find the state $|y^0\rangle$.

Answer: $|y^0\rangle$ must be orthogonal to $|x^0\rangle$ and to $|z^0\rangle$. By inspection we see that

$$|y^0\rangle = \frac{1}{\sqrt{2}}(|z^+\rangle + |z^-\rangle)$$

has the required properties.

(d) A magnetic field acts on the spin creating the Hamiltonian

$$H = -\frac{\Omega}{2} S_z.$$

Express the unitary time development operator $U(t)$ as a matrix in the basis of S_z eigenstates.

Answer:

$$U(t) = e^{-iHt} = \begin{pmatrix} e^{i\Omega t/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\Omega t/2} \end{pmatrix}.$$

(e) At time $t = 0$ the spin is in the state $|x^0\rangle$. Determine the probability that the state is $|x^0\rangle$ at time $t > 0$.

Answer: The state at time t is $|\psi(t)\rangle = U(t)|x^0\rangle$. To obtain the probability we project onto $\langle x^0|$ and then square,

$$\langle x^0|U(t)|x^0\rangle = \frac{1}{2}(e^{i\Omega t/2} + e^{-i\Omega t/2}) = \cos(\Omega t/2), \quad P(t) = |\langle x^0|U(t)|x^0\rangle|^2 = \frac{1}{2}(1 + \cos(\Omega t)).$$

(f) The matrix

$$H_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{pmatrix}$$

with $\omega = e^{2\pi i/3}$ is the analog for qutrit states $\{|0\rangle, |1\rangle, |2\rangle\}$ of the Hadamard transformation on qubit states $\{|0\rangle, |1\rangle\}$,

$$H_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Show that H_3 is unitary.

Answer: The identities $\sum_{k=0}^{2} \omega^k = \sum_{k=0}^{2} \omega^k = 0$ can be applied to show that $H_3^\dagger H_3 = H_3 H_3^\dagger = 1$.

(g) (**hard!**) Alice and Bob share a Bell-like state $|B_{00}\rangle = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}$. Define the matrices acting on the first qutrit

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and show that the 9 distinct states $|\Psi_{xy}\rangle \equiv (R^x T^y \otimes I)|B_{00}\rangle$ form an orthonormal set, with $0 \le x, y \le 2$.

Answer: Consider the expectation value

$$E(x', y', x, y) \equiv \langle \Psi_{x'y'} | \Psi_{xy} \rangle = \langle B_{00} | (|B_{00}\rangle T^{\dagger y'} R^{\dagger x'})(R^x T^y) B_{00}$$

Noting that R and T are unitary, and that $TR = \omega RT$, we find

$$E(x', y', x, y) = \omega^{-y'(x-x')} R^{x-x'} T^{y-y'}.$$

Now consider expectation values $\langle B_{00} | R^{x-x'} T^{y-y'} | B_{00} \rangle$. If $y \neq y$ then $T^{y-y'}$ permutes the basis set for the first qutrit, hence the expectation value vanishes. Setting $y = y'$ we find

$$\langle B_{00} | R^{x-x'} | B_{00} \rangle = \frac{1}{3} \sum_{k=0,2} \omega^{k(x-x')} = \delta_{x,x'}$$

and hence $E(x', y', x, y) = \delta_{x,x'} \delta_{y,y'}$, as claimed.

(h) Alice chooses a unitary transformation to perform on her half of $|B_{00}\rangle$, then gives her qutrit to Bob, who then performs a measurement on the entangled qutrit pair. Explain (in words) the procedure that Alice and Bob can use so that Alice transmits classical information to Bob. You may assume the claim made in part (g) even if you have not proved it. How much classical information (in units of bits) can Alice transmit *via* her half of the qutrit pair.

> **Answer:** Alice can transform B_{00} to the state $B_{xy} = (R^x T^y \otimes I)|B_{00}\rangle$. Bob can then measure to obtain the values of $x, y \in \{0, 1, 2\}$, revealing a unique combination out of 9 possibities. The result is a transfer of $H = \log_2 9 = 3.17$ bits of information transfered by one qutrit. In contrast, regular dense coding transfers 2 bits of information within one qubit.