NAME: ______________________________________

33-658 QCQI Final Exam Dec. 8, 2021

0. (a) Have you completed your FCE, or will you do so? Yes (1 point)/No (0 points)

(b) Did you add comments to the FCE, or will you do so? Yes (2 points)/No (0 points)

1. (70 points) Consider a pair of qubits in the composite Hilbert space $\mathcal{H}_{QM}$. We name the first qubit Q (the "quantum system") and the second M (the "measuring device"). The measuring device has a ready state $|b=0\rangle$ a complementary state $|b=1\rangle$. The measuring qubit will itself be measured in the basis $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. We define a generalized measurement

$$U : |\psi\rangle \otimes |b\rangle \to M_+|\psi\rangle \otimes |+\rangle + (-1)^b M_-|\psi\rangle \otimes |-\rangle$$

where the measurement operators

$$M_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\mathbf{1}, \quad M_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\boldsymbol{\sigma}_3.$$

(a) Show that U is unitary.

(b) Measuring $M = m$ in the $\{|m = \pm\rangle\}$ basis leaves Q in the state $M_m|\psi\rangle$, up to normalization. According to the Born rule, the probability of measurement outcome m is

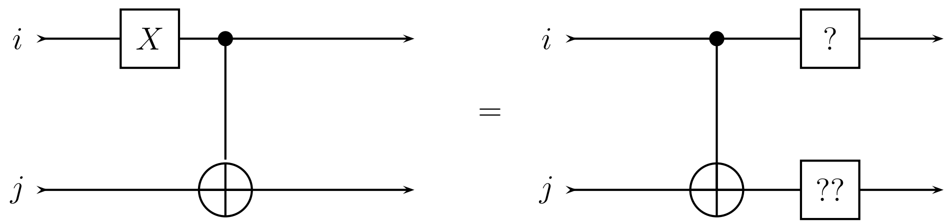
$$P(m) = \langle 0|\langle\psi|U^\dagger \ (I_Q \otimes |m\rangle\langle m|)\, U|\psi\rangle|0\rangle.$$

Evaluate $P(+)$ and the resulting normalized state $|\psi'\rangle$.

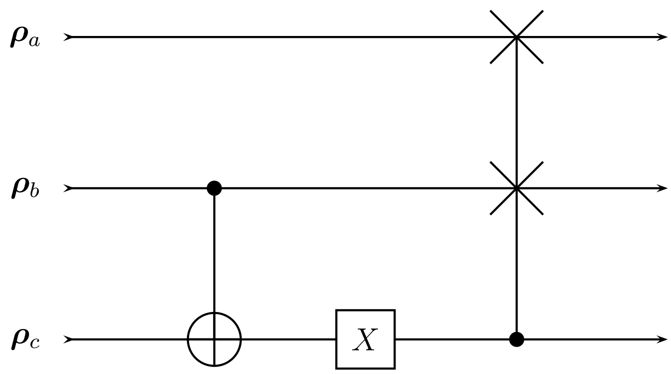
(c) Immediately following the initial measurement that resulted in $M = m$, qubit M is reset to $|0\rangle$ without disturbing qubit Q. A second measurement is performed resulting in outcome m'. Evaluate the conditional probability $P(m' = -|m = +)$.

(d) Consider the initial pure state density operator $\rho_Q = |\psi\rangle\langle\psi|$. The state is measured, as described above, but the measurement outcome is not reported. The new state $\rho'_Q = \mathcal{E}(\rho_Q)$ is given by a mapping of operators. Express this mapping in terms of the operators $M_\pm$.

2. (30 points) Replace the question marks in the following circuit equivalence diagram.



3. (60 points) Consider the circuit below. Each qubit is a spin-1/2 particle in magnetic field at temperature T, with the same initial density operators $\boldsymbol{\rho}$. That is, the initial state is a tensor product of three mixed state density operators. The operations are: controlled not (b is the control for target c); logical `not` on c; a Fredkin gate (controlled swap) that swaps qubits a and b if $c = 1$.



(a) Consider spin a. Its density operator can be written

$$\boldsymbol{\rho}_a = \frac{1}{2}\begin{pmatrix} 1+\eta & 0 \\ 0 & 1-\eta \end{pmatrix}$$

Calculate the thermodynamic entropy $S_\theta(\boldsymbol{\rho})$ as a function of the bias η. How does this vary for small η?

(b) Determine the density operator $\boldsymbol{\rho}'_{bc}$ following the controlled not operation. *i.e.* trace out, or simply disregard, spin a.

(c) Show that, after the controlled not operation, the conditional probability

$$P(b' = 0 | c' = 0) = \frac{(1+\eta)^2}{2(1+\eta^2)}$$

so that the bias of b' (still given $c' = 0$) is

$$\eta'_b = \frac{2\eta}{1+\eta^2}.$$

(d) Following the Fredkin gate, the entropy of a is lower than previously. Explain why this is true, and why this does not violate the second law of thermodynamics. By how much is the entropy of a reduced, in the limit of small η?

4. (40 points) Let $\boldsymbol{\rho}$ be the density operator for a quantum system, and let $\{P_i\}$ be a complete set of orthogonal projectors. Define

$$\boldsymbol{\rho}' = \sum_i P_i \boldsymbol{\rho} P_i.$$

(a) Explain how $\boldsymbol{\rho}'$ relates to a projective measurement.

(b) Show that $S(\boldsymbol{\rho}') = -\mathrm{Tr}\, \boldsymbol{\rho} \log \boldsymbol{\rho}'$ (be careful to distinguish $\boldsymbol{\rho}$ *vs.* $\boldsymbol{\rho}'$). What can this expression for $S(\boldsymbol{\rho}')$ tell us about the change in entropy following a projective measurement?