NAME: SOLUTIONS

33-658 QCQI Final Exam Dec. 8, 2021

1. (70 points)

Consider a pair of qubits in the composite Hilbert space $\mathcal{H}_{QM}$. We name the first qubit Q (the "quantum system") and the second M (the "measuring device"). The measuring device has a ready state $|b=0\rangle$ a complementary state $|b=1\rangle$. The measuring qubit will itself be measured in the basis $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. We define a generalized measurement

$$U : |\psi\rangle \otimes |b\rangle \to M_+|\psi\rangle \otimes |+\rangle + (-1)^b M_-|\psi\rangle \otimes |-\rangle$$

where the measurement operators

$$M_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\mathbf{1}, \quad M_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\boldsymbol{\sigma}_3.$$

(a) Show that U is unitary.

Answer: We must show preservation of inner products. Define $|A\rangle = U|\psi\rangle|b\rangle$ and $\langle B| = \langle b'|\langle\phi|U^\dagger$, and note that

$$M_+^\dagger M_+ = M_-^\dagger M_- = \frac{1}{2}\mathbf{1}.$$

The inner product

$$C = \langle B|A\rangle = \langle\phi|M_+^\dagger M_+|\psi\rangle + \langle\phi|M_-^\dagger M_-|\psi\rangle = \frac{1}{2}\langle\phi|\psi\rangle + \frac{1}{2}(-1)^{b+b'}\langle\phi|\psi\rangle$$

vanishes unless $b = b'$, in which case $C = \langle\phi|\psi\rangle$. Hence inner products are preserved.

(b) Measuring $M = m$ in the $\{|m = \pm\rangle\}$ basis leaves Q in the state $M_m|\psi\rangle$, up to normalization. According to the Born rule, the probability of measurement outcome m is

$$P(m) = \langle 0|\langle\psi|U^\dagger \; (I_Q \otimes |m\rangle\langle m|)\, U|\psi\rangle|0\rangle.$$

Evaluate $P(+)$ and the resulting normalized state $|\psi'\rangle$.

Answer: We have

$$\begin{aligned} P(+) &= \langle 0|\langle\psi|U^\dagger \; (I_Q \otimes |+\rangle\langle +|) \; U|\psi\rangle|0\rangle \\ &= \langle 0|\langle\psi|M_+^\dagger M_+|\psi\rangle|0\rangle = |M_+|\psi\rangle|^2 = \frac{1}{2} \end{aligned}$$

The normalized state is simply $|\psi'\rangle = M_+|\psi\rangle/\sqrt{P(+)} = |\psi\rangle$.

(c) Immediately following the initial measurement that resulted in $M = m$, qubit M is reset to $|0\rangle$ without disturbing qubit Q. A second measurement is performed resulting in outcome m'. Evaluate the conditional probability $P(m' = -|m = +)$.

Answer: The conditional probability

$$P(m' = -|m = +) = \frac{|M_- M_+|\psi\rangle|^2}{P(+)} = \frac{1}{2}$$

Since the measurement states are not orthogonal, the measurement outcomes are not exclusive.

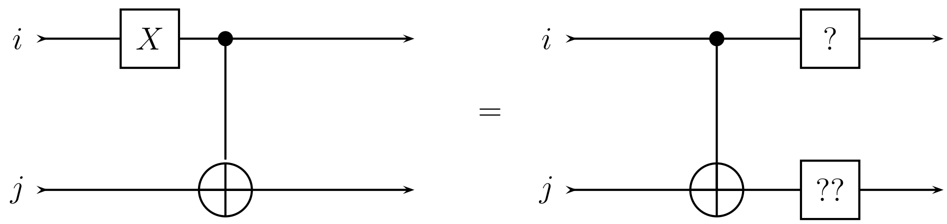
(d) Consider the initial pure state density operator $\rho_Q = |\psi\rangle\langle\psi|$. The state is measured, as described above, but the measurement outcome is not reported. The new state $\rho_Q' = \mathcal{E}(\rho_Q)$ is given by a mapping of operators. Express this mapping in terms of the operators $M_\pm$.

Answer: The mapping

$$\mathcal{E}(\boldsymbol{\rho}_Q) = \sum_m M_m \boldsymbol{\rho}_Q M_m^\dagger$$

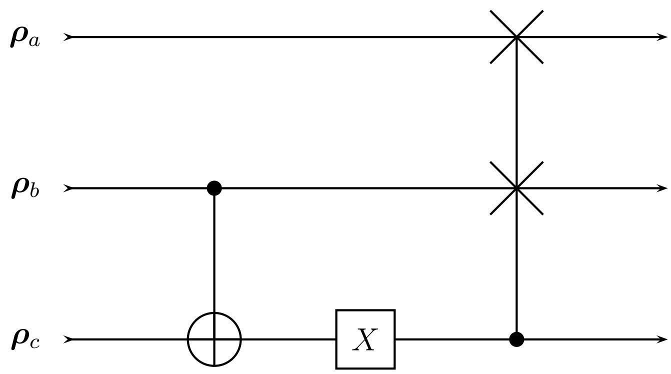
yields the correct states with the correct probabilities. That is, the set $\{M_\pm\}$ are the Kraus operators for the mapping.

2. (30 points) Replace the question marks in the following circuit equivalence diagram.



Answer: Define the `cnot` gate $\boldsymbol{C}_{ij}|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle$ where addition is taken mod 2. Then $\boldsymbol{C}_{ij}X_i|x\rangle|y\rangle = |x \oplus 1\rangle|x \oplus 1 \oplus y\rangle = \boldsymbol{X}_i\boldsymbol{X}_j\boldsymbol{C}_{ij}|x\rangle|y\rangle$. Thus each gate ? and ?? is $\boldsymbol{X}$.

3. (60 points) Consider the circuit below. Each qubit is a spin-1/2 particle in magnetic field at temperature T, with the same initial density operators $\boldsymbol{\rho}$. That is, the initial state is a tensor product of three mixed state density operators. The operations are: controlled not (b is the control for target c); logical `not` on c; a Fredkin gate (controlled swap) that swaps qubits a and b if $c = 1$.



(a) Consider spin a. Its density operator can be written

$$\boldsymbol{\rho}_a = \frac{1}{2}\begin{pmatrix} 1+\eta & 0 \\ 0 & 1-\eta \end{pmatrix}$$

Calculate the thermodynamic entropy $S_\theta(\boldsymbol{\rho})$ as a function of the bias η. How does this vary for small η?

answer: Summing over eigenvalues of $\boldsymbol{\rho}$, and setting $k_\mathrm{B} = 1$,

$$S_\theta = -\sum_k \rho_{kk} \ln \rho_{kk} = -\frac{1}{2}(1+\eta)\ln\left(\frac{1}{2}(1+\eta)\right) - \frac{1}{2}(1-\eta)\ln\left(\frac{1}{2}(1-\eta)\right).$$

For small η this varies as $S = \ln 2 - \eta^2/2 + \ldots$.

(b) Determine the density operator $\boldsymbol{\rho}'_{bc}$ following the controlled not operation. *i.e.* trace out, or simply disregard, spin a.

Answer: First, note the product $(1+\eta)(1-\eta) = 1-\eta^2$. Expressing the density operator as a matrix in the basis as $\{|bc\rangle = |00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$\boldsymbol{\rho}_{bc} = (1/4)\ \mathbf{diag}[(1+\eta)^2, (1-\eta^2), (1-\eta^2), (1-\eta)^2]$$

transforms to

$$\boldsymbol{\rho}'_{bc} = (1/4)\ \mathbf{diag}[(1+\eta)^2, (1-\eta^2), (1-\eta)^2, (1-\eta^2)].$$

Note that the last two entries were interchanged by the `not` operation when $b = 1$.

(c) Show that, after the controlled not operation, the conditional probability

$$P(b' = 0|c' = 0) = \frac{(1+\eta)^2}{2(1+\eta^2)}$$

so that the bias of b' (still given $c' = 0$) is

$$\eta'_b = \frac{2\eta}{1+\eta^2}.$$

Answer: First we work out the joint and marginal probabilities

$$P(b' = 0, c' = 0) = \langle 00|\boldsymbol{\rho}'_{bc}|00\rangle = \frac{1}{4}(1+\eta)^2,$$

$$P(c' = 0) = \langle 00|\boldsymbol{\rho}'_{bc}|00\rangle + \langle 10|\boldsymbol{\rho}'_{bc}|10\rangle = \frac{1}{2}(1+\eta^2),$$

The we evaluate the conditional probability

$$P(b' = 0|c' = 0) = \frac{P(b' = 0, c' = 0)}{P(c' = 0)} = \frac{(1+\eta)^2}{2(1+\eta^2)}$$

and the bias

$$\eta'_b = 2P(b' = 0|c' = 0) - 1 = \frac{2\eta}{1+\eta^2}.$$

(d) Following the Fredkin gate, the entropy of a is lower than previously. Explain why this is true, and why this does not violate the second law of thermodynamics. By how much is the entropy of a reduced, in the limit of small η?

Answer: Owing to the `not` gate, the controlled Fredkin gate swaps a and b when $c' = 0$. Since $\eta'_b \approx 2\eta$, this swap doubles the bias of a, hence reducing the entropy of a. This does not violate the second law because in that case, the entropy of b *increases* to the same degree that the entropy of a decreases. Since $c' = 0$ with probability 1/2 (to lowest order), then the average increase in η_a is $\eta'_a = 3\eta_a/2$, and the entropy drops by $(9-4)\eta^2/8 = 5\eta^2/8$.

This technique for enhancing the bias of a spin is known as "Algorithmic Cooling" and was introduced by Fernandez, Lloyd, Mor and Roychowdhury in 2004.

4. (40 points) Let $\boldsymbol{\rho}$ be the density operator for a quantum system, and let $\{P_i\}$ be a complete set of orthogonal projectors. Define

$$\boldsymbol{\rho}' = \sum_i P_i \boldsymbol{\rho} P_i.$$

(a) Explain how $\boldsymbol{\rho}'$ relates to a projective measurement.

Answer: The density operator $\boldsymbol{\rho}'$ is the quantum state following a projective measurement in which we remain unaware of the measurement outcome. We know the quantum state is an eigenstate of P_i with probability $\langle i|P_i|i\rangle$, but we do not know the value of i.

(b) Show that $S(\boldsymbol{\rho}') = -\text{Tr}\, \boldsymbol{\rho} \log \boldsymbol{\rho}'$ (be careful to distinguish $\boldsymbol{\rho}$ *vs.* $\boldsymbol{\rho}'$). What can this expression for $S(\boldsymbol{\rho}')$ tell us about the change in entropy following a projective measurement?

Answer: We will evaluate the trace using the basis set projector eigenstates $\{|i\rangle\}$. Note that

$$\log \boldsymbol{\rho}' = \sum_i P_i \log \rho_i', \quad \rho_i' = \langle i|\boldsymbol{\rho}|i\rangle, \quad P_i \log \boldsymbol{\rho}'|i\rangle = \log \boldsymbol{\rho}'|i\rangle.$$

Expanding the operators and invoking orthogonality,

$$\begin{aligned} S(\boldsymbol{\rho}') &= -\sum_i \langle i| \left(\sum_j P_j \boldsymbol{\rho} P_j \sum_k P_k \log \boldsymbol{\rho}' \right) |i\rangle \\ &= -\sum_i \langle i| \left(\boldsymbol{\rho}\ P_i \log \boldsymbol{\rho}'\right) |i\rangle \\ &= -\sum_i \langle i|\boldsymbol{\rho} \log \boldsymbol{\rho}'|i\rangle = -\text{Tr}\ \boldsymbol{\rho} \log \boldsymbol{\rho}'. \end{aligned}$$

Recall the relative entropy is non-negative,

$$D(\boldsymbol{\rho}'||\boldsymbol{\rho}) = -\text{Tr}\ \boldsymbol{\rho} \log \boldsymbol{\rho}' + \text{Tr}\ \boldsymbol{\rho} \log \boldsymbol{\rho} = S(\boldsymbol{\rho}') - S(\boldsymbol{\rho}) \geq 0.$$

Hence a projective measurement *increases* the entropy.