NAME: ___________________________________

33-658 Quantum I — Midterm Exam — Oct. 11, 2022

This exam consists of four questions, each with multiple parts worth 10 points. Some parts are quite easy, and some can be answered independently of others. You might find these Pauli matrices to be useful.

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. In a z-oriented Stern-Gerlach experiment, a spin $|z^+\rangle$ in the input channel $|i\rangle$ is in the product state $|i, z^+\rangle = |i\rangle|z^+\rangle$. While passing through the apparatus, it is deflected upwards into the state $|u, z^+\rangle$, while $|i, z^-\rangle$ is deflected downwards into the state $|d, z^-\rangle$.

a) If the initial state is $|i, x^+\rangle$, what is the final state after passing through the Stern Gerlach apparatus.

b) Is the final state a product state or is it entangled? If a product express it as a product; if entangled show that it cannot be expressed as a product.

2. Suppose you are given a qubit in an unknown quantum state with density operator

$$\rho = \frac{1}{2}\left(I + a_x X + a_y Y + a_z Z\right)$$

a) Evaluate $\mathrm{Tr}\left(\rho X\right)$.

b) Suppose you had many identical copies of ρ (e.g. from repetitions of an experiment that creates ρ). How would you determine ρ?

3. The IBM quantum computer has a limited set of gates that operate on a single qubit $|\psi\rangle = \sqrt{1-p}\,|0\rangle + \sqrt{p}\,e^{i\varphi}|1\rangle$. It has logical "not" (symbolized X), Hadamard (symbolized H), the phase gate

$$P(\theta) = e^{i\theta/2} e^{-i(\theta/2)Y}$$

that advances the phase φ by θ, and the gate

$$RY(\theta) = e^{-i(\theta/2)Y}$$

that rotates the Bloch sphere around the y axis by θ. It also has a destructive measurement operator M_z that reveals the value of the state in the computational (0/1) basis.

a) Draw a circuit to measure $|\psi\rangle$ in the $|\pm\rangle$ basis. Specify which value of M_z corresponds to each value of $\pm$.

b) In our test of the Bell inequality we measured in a basis W that lay half-way between X and Z. Write down an operator corresponding to this observable and draw a circuit to achieve this measurement. Specify which value of M_z corresponds to each value of $\pm$ in the $|w^{\pm}\rangle$ basis.

4. Alice and Bob share the Bell state

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$$

a) Alice measures in the $|x_a^{\pm}\rangle$ basis. Evaluate the partial inner product $\langle x_a^{+}|B_{11}\rangle$.

b) Calculate the probability that Alice obtains the result $|x_a^{+}\rangle$

c) Write down the density operator ρ_{ab} for the Bell state. You can express it as a sum of dyads or as a matrix. If you choose to write a matrix be sure to completely specify the basis set you are using.

d) Evaluate the partial trace $\mathrm{Tr}_b\ \rho_{ab}$.