NAME: SOLUTIONS

33-658 Quantum I | Midterm Exam | Oct. 11, 2022

This exam consists of four questions, each with multiple parts worth 10 points. Some parts are quite easy, and some can be answered independently of others. You might find these Pauli matrices to be useful.

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1. In a z-oriented Stern-Gerlach experiment, a spin $|z^+\rangle$ in the input channel $|i\rangle$ is in the product state $|i, z^+\rangle = |i\rangle|z^+\rangle$. While passing through the apparatus, it is deflected upwards into the state $|u, z^+\rangle$, while $|i, z^-\rangle$ is deflected downwards into the state $|d, z^-\rangle$.

a) If the initial state is $|i, x^+\rangle$, what is the final state after passing through the Stern Gerlach apparatus.

Answer: Expressing $|i, x^+\rangle$ as $(|i, z^+\rangle + |i, z^-\rangle)/\sqrt{2}$, we obtain the final state $|\Psi\rangle = (|u, z^+\rangle + |d, z^-\rangle)/\sqrt{2}$.

b) Is the final state a product state or is it entangled? If a product express it as a product; if entangled show that it cannot be expressed as a product.

Answer: It is entangled. One way to show this is to note that the spin state for upwards deflection differs from the spin state for downwards deflection, so the deflection and spin states cannot be factored. Alternatively, express

$$|\Psi\rangle = \sum_{jk} \Psi_{jk}|j\rangle|k\rangle$$

where j runs over u, d, and k runs over z^+, z^-. Note that Ψ_{jk} has rank greater than one.

2. Suppose you are given a qubit in an unknown quantum state with density operator

$$\rho = \frac{1}{2}\left(I + a_x X + a_y Y + a_z Z\right)$$

a) Evaluate Tr $\big(\rho X\big)$.

Answer: The products YX and ZX are traceless, while $X^2 = I$ so thatTr $X^2 = 2$. Hence Tr $\rho X = a_x$.

b) Suppose you had many identical copies of ρ (e.g. from repetitions of an experiment that creates ρ). How would you determine ρ?

Answer: Tr $\big(\rho X\big)$ is the expectation value of the observable X. Repeated measurements of X on equivalent states will yield the average $\langle X\rangle = \text{Tr}\ \big(\rho X\big) = a_x$. Similarly, repeated measurements of Y and Z yield a_y and a_z. This process is known as tomographic state reconstruction.

3. The IBM quantum computer has a limited set of gates that operate on a single qubit $|\psi\rangle = \sqrt{1-p}\ |0\rangle + \sqrt{p}\ e^{i\varphi}|1\rangle$. It has logical "not" (symbolized X), Hadamard (symbolized H), the phase gate

$$P(\theta) = e^{i\theta/2}e^{-i(\theta/2)Y}$$

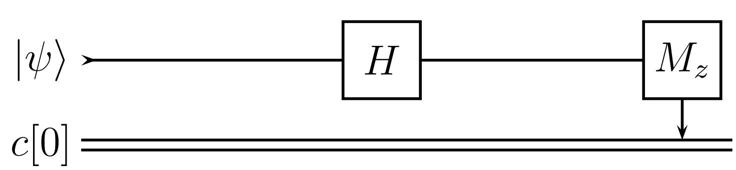
that advances the phase φ by θ, and the gate

$$RY(\theta) = e^{-i(\theta/2)Y}$$

that rotates the Bloch sphere around the y axis by θ. It also has a destructive measurement operator M_z that reveals the value of the state in the computational (0/1) basis.

a) Draw a circuit to measure $|\psi\rangle$ in the $|\pm\rangle$ basis. Specify which value of M_z corresponds to each value of $\pm$.

Answer: The Hadamard gate will transform $|+\rangle = (|z^+\rangle + |z^-\rangle)/\sqrt{2}$ into $|z^+\rangle = |0\rangle$ (reported as $c[0] = 0$). Similarly $|-\rangle$ transforms into $|z^-\rangle = |1\rangle$ (reported as $c[0] = 1$).

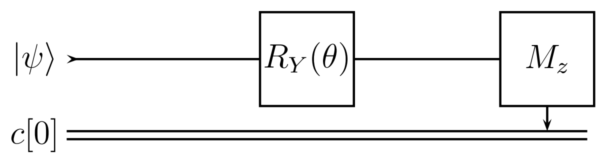


b) In our test of the Bell inequality we measured in a basis W that lay half-way between X and Z. Write down an operator corresponding to this observable and draw a circuit to achieve this measurement. Specify which value of M_z corresponds to each value of $\pm$ in the $|w^{\pm}\rangle$ basis.

Answer: In the standard $|z^{\pm}\rangle$ basis $R_y(\theta)$ has the representation

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}.$$

Note that positive θ rotates $|z^+\rangle$ towards $|w^+\rangle$. We need the opposite, so we use the gate $R_y(-\frac{\pi}{4})$. This will transform $|w^+\rangle$ into $|z^+\rangle$ (reported as $c[0] = 0$). Similarly $|w^-\rangle$ transforms into $|z^-\rangle$ (reported as $c[0] = 1$).



4. Alice and Bob share the Bell state

$$|B_{11}\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big)$$

a) Alice measures in the $|x_a^{\pm}\rangle$ basis. Evaluate the partial inner product $\langle x_a^+|B_{11}\rangle$.

Answer:

$$\langle x_a^+|B_{11}\rangle = \frac{1}{\sqrt{2}}\big(\langle 0_a| + \langle 1_a|\big)\frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle\big) = \frac{1}{2}\big(|1_b\rangle - |0_b\rangle\big),$$

b) Calculate the probability that Alice obtains the result $|x_a^+\rangle$

Answer:

$$P(x_a^+) = |\langle x_a^+|B_{11}\rangle|^2 = \frac{1}{4}\big(\langle 1_b| - \langle 0_b|\big)\big(|1_b\rangle - |0_b\rangle\big) = \frac{1}{2}$$

c) Write down the density operator ρ_{ab} for the Bell state. You can express it as a sum of dyads or as a matrix. If you choose to write a matrix be sure to completely specify the basis set you are using.

Answer: As a sum of operators,

$$\rho_{ab} = |B_{11}\rangle\langle B_{11}| = \frac{1}{2}\big(|01\rangle\langle 01| - |10\rangle\langle 01| - |01\rangle\langle 10| + |10\rangle\langle 10|\big).$$

As a matrix we will take the basis set $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ (in that order). Then

$$\rho_{ab} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

d) Evaluate the partial trace $\mathrm{Tr}_b\ \rho_{ab}$.

Answer:

$$\rho_a = \mathrm{Tr}_b\ \rho_{ab} = \sum_b \langle 0_b|\rho_{ab}|0_b\rangle = \frac{1}{2}\big((|0_a\rangle\langle 0_a| + |1_a\rangle\langle 1_a|\big) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note we could use this to answer part (b) by forming the projector

$$\Pi_{x_a^+} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

then evaluating

$$P(x_a^+) = \mathrm{Tr}\ \big(\Pi_{x_a^+}\rho_a\big) = \frac{1}{2}.$$