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33-658 Quantum I Final Exam December 2022

This is a take-home exam. You may use any resources (book, notes, WWW, etc.) that you wish other than discussing with another person. When you are finished please scan to a clear black&white PDF and upload to Canvas.

0. (a) Have you completed your FCE, or will you do so? Yes (1 point)/No (0 points)

(b) Did you add comments to the FCE, or will you do so? Yes (2 points)/No (0 points)

1. (15 points) A random variable Y depends conditionally on the random variable X, and Z depends on Y but not on X. Hence the joint probability factors as $p_{XYZ}(x, y, z) = p_X(x)p_{XY}(y|x)p_{YZ}(z|y)$. Prove the bound on classical mutual information, $I(X; Y) \geq I(X; Z)$.

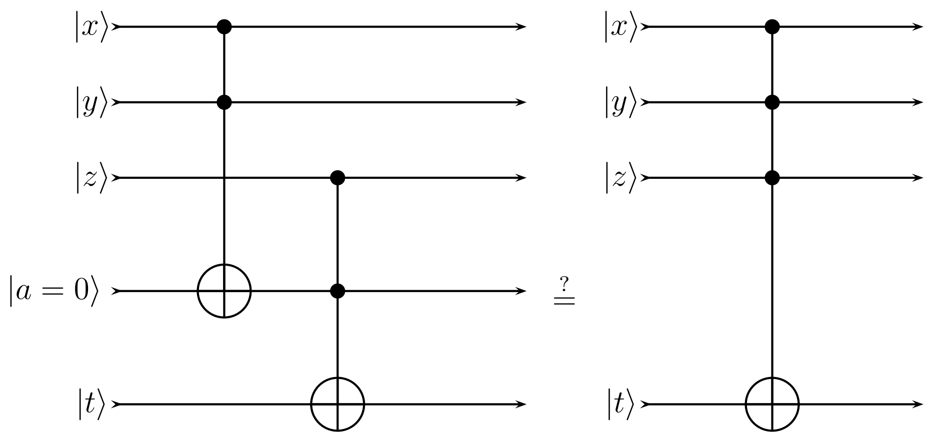
2. (15 points) Given pure states $\boldsymbol{\rho}_1 = |\psi\rangle\langle\psi| = (1 + \vec{a_1} \cdot \vec{\boldsymbol{\sigma}})/2$ and $\boldsymbol{\rho}_2 = |\phi\rangle\langle\phi| = (1 + \vec{a_2} \cdot \vec{\boldsymbol{\sigma}})/2$, with $\langle\psi|\phi\rangle = \cos(\theta)$, show that $\vec{a}_1 \cdot \vec{a}_2 = \cos(2\theta)$.

3. Consider the Hamiltonian $H = \mathcal{E}N$ with $N = a^\dagger a$ the number operator. Such a Hamiltonian has the spectrum $E_n = n\,\mathcal{E}$ with n taking non-negative integer values. It is equivalent to a harmonic oscillator, neglecting the zero point energy.

(a) (20 points) Determine the equilibrium state ω and its free energy.

(b) (20 points) Calculate its thermodynamic energy E and entropy S.

4. A Toffoli gate is a doubly-controlled-not gate that takes the "logical and" of the two control bit inputs, x and y, as the control of the target bit t. The final target bit is replaced by $t \oplus xy$. I wish to create a triply-controlled-not to replace the target bit by $t \oplus xyz$ by first storing the product of x and y in an ancillary qubit a, then repeating the process with z and a controlling the target t. The circuit on the left is my attempt to mimic the triply-controlled-not gate shown on the right.



(a) (15 points) Calculate the final state of each circuit assuming the inputs, x, y, z, and t, take binary (0/1) values.

(b) (15 points) Calculate the final states again, replacing the bit value x with the superposition state $|\psi_x\rangle = \alpha|0\rangle + \beta|1\rangle$.

(c) (20 points) Imagine that after executing my circuit I measure the ancillary qubit and obtain $a = 1$. What is the conditional state following the measurement? Explain why this state differs from the proper output of the triply-controlled-not gate.

(d) (20 points) A single additional operation can repair my circuit. What is it, and why does it work?

5. In Schroedinger's cat paradox, a process is described that places a cat into a superposition state of alive and dead, $|\psi\rangle = (|a\rangle + |d\rangle)/\sqrt{2}$. The cat and the entire aparatus is hidden inside a box so that we cannot see it.

a) (15 points) Write down the density matrix for this state, ρ.

b) (15 points) The cat interacts with its environment, generating random phase flips. Derive and solve Lindblad's differential equation for this process. What is the final steady state?

c) (15 points) Evaluate the initial and final state entropies.

d) (15 points) What physical interpretation would you assign to the final state? What would the state be if you opened the box and looked inside?