NAME: SOLUTIONS

33-658 Quantum I Final Exam December 2022

This is a take-home exam. You may use any resources (book, notes, WWW, etc.) that you wish other than discussing with another person. When you are finished please scan to a clear black&white PDF and upload to Canvas.

0. (a) Have you completed your FCE, or will you do so? **Yes** (1 point)/No (0 points)

(b) Did you add comments to the FCE, or will you do so? **Yes** (2 points)/No (0 points)

1. (15 points) A random variable Y depends conditionally on the random variable X, and Z depends on Y but not on X. Hence the joint probability factors as $p_{XYZ}(x, y, z) = p_X(x)p_{XY}(y|x)p_{YZ}(z|y)$. Prove the bound on classical mutual information, $I(X;Y) \geq I(X;Z)$.

Answer: Apply the chain rule to the mutual information between X and the joint value of (Y, Z),

$$I(X;(Y,Z)) = I(X;Z) + I(X;Y|Z) = I(X;Y) + I(X;Z|Y)$$

Since $I(X;Z|Y) = 0$, and $I(X;Y|Z) \geq 0$ then $I(X;Y) \geq I(X;Z)$. The shows that the information that measurements Y provide about the property X cannot be increased by further post-processing that converts Y into Z. This is the *data processing inequality*; no transformation of data Y can increase the information that Z contains regarding X.

2. (15 points) Given pure states $\boldsymbol{\rho}_1 = |\psi\rangle\langle\psi| = (1 + \vec{a_1} \cdot \vec{\boldsymbol{\sigma}})/2$ and $\boldsymbol{\rho}_2 = |\phi\rangle\langle\phi| = (1 + \vec{a_2} \cdot \vec{\boldsymbol{\sigma}})/2$, with $\langle\psi|\phi\rangle = \cos(\theta)$, show that $\vec{a}_1 \cdot \vec{a}_2 = \cos(2\theta)$.

Answer This is adapted from Schumacher & Westmoreland problem #8.1. From exercise 8.31 we have Tr $\boldsymbol{\rho}_1\boldsymbol{\rho}_2 = (1 + \vec{a}_1 \cdot \vec{a}_2)/2$ as can be proved using properties of Pauli matrices. Now consider

$$
\begin{aligned}
Tr\ \boldsymbol{\rho}_1\boldsymbol{\rho}_2 &= \sum_k \langle k|\psi\rangle\langle\psi|\phi\rangle\langle\phi|k\rangle \\
&= \sum_k \langle\psi|\phi\rangle\langle\phi|k\rangle\langle k|\psi\rangle \\
&= |\langle\phi|\psi\rangle|^2 \\
&= \frac{1}{2}(1 + \cos(2\theta))
\end{aligned}
$$

3. Consider the Hamiltonian $H = \mathcal{E}N$ with $N = a^\dagger a$ the number operator. Such a Hamiltonian has the spectrum $E_n = n\ \mathcal{E}$ with n taking non-negative integer values. It is equivalent to a harmonic oscillator, neglecting the zero point energy.

(a) (20 points) Determine the equilibrium state ω and its free energy.

Answer: The equilibrium state density operator

$$\omega = \frac{1}{Z}\ e^{-H/k_{\mathrm{B}}T}$$

where the partition function normalizes the equilibrium state density operator with the value

$$Z = \mathrm{Tr}\ e^{-H/k_{\mathrm{B}}T} = \sum_{n=0}^{\infty} e^{-n\mathcal{E}/k_{\mathrm{B}}T} = \frac{1}{1 - e^{-\mathcal{E}/k_{\mathrm{B}}T}}.$$

We get the free energy from the partition function as

$$F(\omega HT) = -k_{\mathrm{B}}T\ \ln Z = k_{\mathrm{B}}T\ \ln(1 - e^{-\mathcal{E}/k_{\mathrm{B}}T}).$$

(b) (20 points) Calculate its thermodynamic energy E and entropy S.

Answer: The energy

$$E = \mathrm{Tr}\ \omega H = \frac{1}{Z}\ \sum_n n\ \mathcal{E}\ e^{-n\mathcal{E}/k_{\mathrm{B}}T}$$

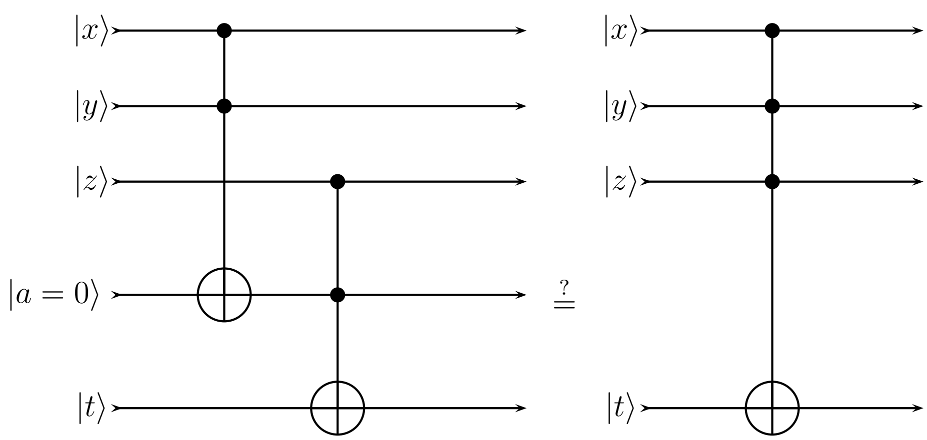
This can be evaluated as

$$E = -\frac{\mathcal{E}}{Z}\frac{\partial}{\partial(\mathcal{E}/k_{\rm B}T)}Z = \frac{\mathcal{E}e^{-\mathcal{E}/k_{\rm B}T}}{1-e^{-\mathcal{E}/k_{\rm B}T}} = \frac{\mathcal{E}}{e^{\mathcal{E}/k_{\rm B}T}-1}.$$

Entropy can be calculated from $S = -k_{\rm B}{\rm Tr}\ \omega\ \ln\omega$, from $S = -\partial F/\partial T$, or simply as

$$S = (E-F)/T = \frac{\mathcal{E}}{T(e^{\mathcal{E}/k_{\rm B}T}-1)} - k\ln\left(1-e^{-\mathcal{E}/k_{\rm B}T}\right)$$

4. A Toffoli gate is a doubly-controlled-not gate that takes the "logical and" of the two control bit inputs, x and y, as the control of the target bit t. The final target bit is replaced by $t\oplus xy$. I wish to create a triply-controlled-not to replace the target bit by $t\oplus xyz$ by first storing the product of x and y in an ancillary qubit a, then repeating the process with z and a controlling the target t. The circuit on the left is my attempt to mimic the triply-controlled-not gate shown on the right.



(a) (15 points) Calculate the final state of each circuit assuming the inputs, x, y, z, and t, take binary (0/1) values.

Answer: The mimic circuit results in the final state

$$|x\rangle|y\rangle|z\rangle|xy\rangle|t\oplus xy\rangle$$

while the desired circuit yields

$$|x\rangle|y\rangle|z\rangle|t\oplus xy\rangle.$$

(b) (15 points) Calculate the final states again, replacing the bit value x with the superposition state $|\psi_x\rangle = \alpha|0\rangle + \beta|1\rangle$.

Answer: The mimic circuit results in the final state

$$\alpha|0\rangle|y\rangle|z\rangle|0\rangle|t\rangle + \beta|1\rangle|y\rangle|z\rangle|y\rangle|t \oplus yz\rangle$$

while the desired circuit yields

$$\alpha|0\rangle|y\rangle|z\rangle|t\rangle + \beta|1\rangle|y\rangle|z\rangle|t \oplus yz\rangle$$

Note that the target bit has become entangled with the ancillary bit.

(c) (20 points) Imagine that after executing my circuit I measure the ancillary qubit and obtain $a = 1$. What is the conditional state following the measurement? Explain why this state differs from the proper output of the triply-controlled-not gate.

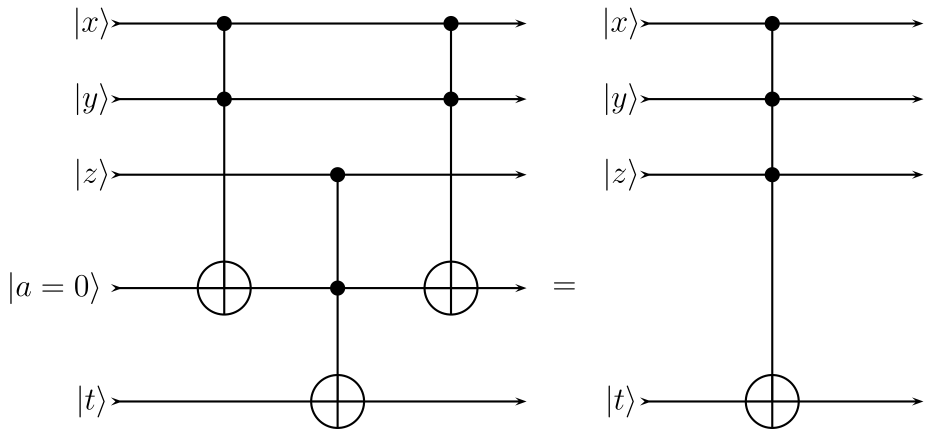
Answer: Since $a = 1$, we may assert that $x = 1$ and $y = 1$ even though x was (and y could have been) in a superposition state. The conditional state of the mimic circuit is

$$|1\rangle|1\rangle|z\rangle|1\rangle|t \oplus z\rangle,$$

which lacks dependence on the x and y states.

(d) (20 points) A single additional operation can repair my circuit. What is it, and why does it work?

Answer: The following page shows a working circuit that "uncomputes" the ancillary bit and thereby removes its entanglement with the target bit. This is the reason that many quantum circuits appear as mirror images containing superficially unnecessary final operations.


$|x\rangle$
$|y\rangle$
$|z\rangle$
$|a = 0\rangle$
$|t\rangle$
$=$
$|x\rangle$
$|y\rangle$
$|z\rangle$
$|t\rangle$

5. In Schroedinger's cat paradox, a process is described that places a cat into a superposition state of alive and dead, $|\psi\rangle = (|a\rangle + |d\rangle)/\sqrt{2}$. The cat and the entire aparatus is hidden inside a box so that we cannot see it.

a) (15 points) Write down the density matrix for this state, ρ.

Answer:

$$\rho_i = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

b) (15 points) The cat interacts with its environment, generating random phase flips. Derive and Lindblad's differential equation for this process. What is the final steady state?

Answer: This the phase flipping example that we worked out in class. The Lindblad operator is $L = \Lambda Z$ yielding the Lindblad equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \Lambda^2 \left(Z\rho Z - \rho \right)$$

In components, we find that the state populations ρ_{aa} and ρ_{dd} are constants, while the coherences ρ_{ad} and ρ_{da} vanish exponentially at the rate $|\Lambda|^2$. The final state is

$$\rho_f = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

c) (15 points) Evaluate the initial and final state entropies.

Answer: The initial state eigenvalues are $\{0, 1\}$ with entropy $S(\rho_i) = -\log 1 - 0$ $= 0$. The final state eigenvalues are $\{1/2, 1/2\}$ with entropy $S(\rho_f) = -(1/2)\log(1/2)$ $-(1/2)\log(1/2) = \log 2$.

d) (15 points) What physical interpretation would you assign to the final state? What would the state be if you opened the box and looked inside?

Answer: This is a mixed state in which the cat has a 50-50% chance of being either alive or dead, but not both. The probabilities reflect our lack of knowledge of the final state. If we opened the box we would observe one state or the other.