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33-658 QCQI Midterm Exam Oct. 12, 2023

This exam has two questions each with multiple parts, some of which can be solved independently of others. When asked to show (or to evaluate, calculate, *etc.*), please show your work in the space provided. When a question is posed, answer in words with one or two clear and concise sentences explaining your answer. Use the back of your paper as needed. Here are some useful formulas:

Pauli operators and their matrices:

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\cos^2\theta + \sin^2\theta = 1, \quad \cos^2\theta = (1+\cos(2\theta))/2, \quad \sin^2\theta = (1-\cos(2\theta))/2$$

Beam splitter actions:

$$|a\rangle \to \frac{1}{\sqrt{2}}\left(|c\rangle + |d\rangle\right), \quad |b\rangle \to \frac{1}{\sqrt{2}}\left(-|c\rangle + |d\rangle\right)$$

1. Exercises with density operators.

Let $\hat{\psi} = (\psi_x, \psi_y, \psi_z)$ be a unit vector. Recall that $\rho_\psi = (1 + \hat{\psi}\cdot\vec{\sigma})/2$ is a pure state density operator representing the state $|\psi\rangle$ corresponding to spin up in the direction $\hat{\psi}$. Define $\mathbf{S}_{\hat{\psi}} \equiv \hat{\psi}\cdot\mathbf{S}$.

(a) Show that $\mathbf{S}_{\hat{\psi}}\rho_\psi = \rho_\psi$.

(b) Evaluate Tr $\rho_\psi \rho_\phi$ in terms of the components of $\hat{\psi}$ and $\hat{\phi}$, and also in terms of the angle θ between $\hat{\psi}$ and $\hat{\phi}$.

(c) Evaluate the expectation value of $\mathbf{S}_{\hat{\psi}}$ in the state ρ_ϕ.

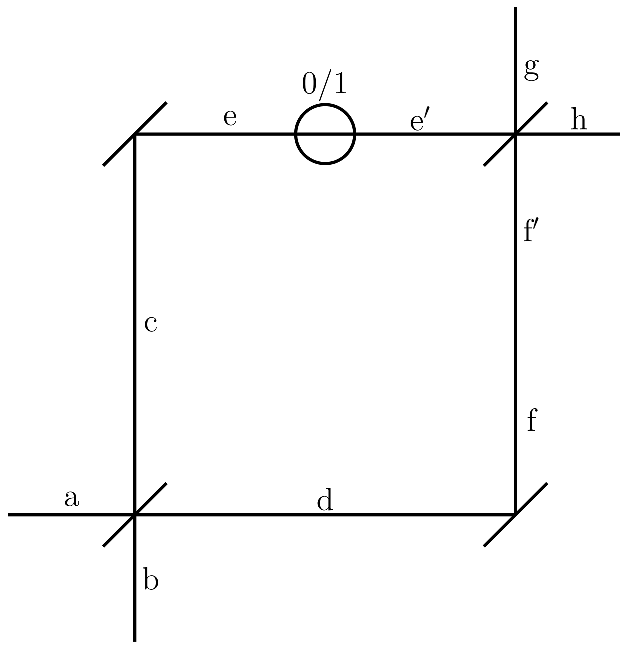
(d) A pure state density operator evolves in time as

$$\rho(t) = U(t)\rho(0)U^\dagger(t).$$

How would you expect a mixed state density operator to evolve? How would Tr $\rho_\psi \rho_\phi$ evolve?

2. Consider the Mach-Zehnder interferometer shown below. An atom is placed on the arm $e \to e'$. Each letter (including primes) represents a possible state for a photon as it traverses the device. The atom may be in its ground state 0 or its excited state 1. We will model the atom-photon interaction as

$$|e\rangle|0\rangle \to |e'\rangle|1\rangle.$$



(a) What is the dimension of the Hilbert space for the composite photon/atom system?

(b) Assume a photon enters in state $|a\rangle$ while the atom is in its ground state $|0\rangle$. Determine the final state of the system $|\Psi\rangle$ as the photon exits the interferometer in arm g or h.

(c) Calculate the conditional probability that the atom is in its excited state given that the photon exits through arm g.

(d) What additional condition must be imposed on our model of the atom-photon interaction to ensure unitarity?

(e) Consider a new atom-photon interaction model in which the atom in its ground state becomes excited, but it absorbs the photon in the process. For clarity, we will denote the state of "no photon" by $|n\rangle$. If the atom is in its excited state it remains excited and the photon passes through unaffected. Specifically,

$$|e\rangle|0\rangle \to |n\rangle|1\rangle, \quad |e\rangle|1\rangle \to |e'\rangle|1\rangle.$$

Repeat your calculation of part (b) for the final state according to this new model.

(f) Repeat your calculation of part (c) under the new model. How (if at all) would your result change if the initial state of the atom were unknown?