NAME: SOLUTIONS

33-658 QCQI Midterm Exam Oct. 12, 2023

This exam has two questions each with multiple parts, some of which can be solved independently of others. When asked to show (or to evaluate, calculate, *etc.*), please show your work in the space provided. When a question is posed, answer in words with one or two clear and concise sentences explaining your answer. Use the back of your paper as needed. Here are some useful formulas:

Pauli operators and their matrices:

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\cos^2\theta + \sin^2\theta = 1, \quad \cos^2\theta = (1+\cos(2\theta))/2, \quad \sin^2\theta = (1-\cos(2\theta))/2$$

Beam splitter actions:

$$|a\rangle \to \frac{1}{\sqrt{2}}\left(|c\rangle + |d\rangle\right), \quad |b\rangle \to \frac{1}{\sqrt{2}}\left(-|c\rangle + |d\rangle\right)$$

1. Exercises with density operators.

Let $\hat{\psi} = (\psi_x, \psi_y, \psi_z)$ be a unit vector. Recall that $\rho_\psi = (1 + \hat{\psi}\cdot\vec{\sigma})/2$ is a pure state density operator representing the state $|\psi\rangle$ corresponding to spin up in the direction $\hat{\psi}$, and similarly for $\hat{\phi}$. Define $\mathbf{S}_{\hat{\psi}} \equiv \hat{\psi}\cdot\mathbf{S}$.

(a) Show that $\mathbf{S}_{\hat{\psi}}\rho_\psi = \rho_\psi$.

**Answer:** The simplest answer is to express $\rho_\psi = |\psi\rangle\langle\psi|$, then

$$\left(\hat{\psi}\cdot\mathbf{S}\right)\rho_\psi = \left(\hat{\psi}\cdot\mathbf{S}|\psi\rangle\right)\langle\psi| = \left(+1\ |\psi\rangle\right)\langle\psi| = \rho_\psi.$$

Alternatively, one can express

$$\rho_\psi = \frac{1}{2}\left(1 + \psi_x X + \psi_y Y + \psi_z Z\right)$$

and then use identities such as $X^2 = 1$ and $XY = -YX$ to simplify the product of $\left(\hat{\psi}\cdot\mathbf{S}\right)$ with $\rho_\psi$.

(b) Evaluate Tr $\rho_\psi \rho_\phi$ in terms of the components of $\hat{\psi}$ and $\hat{\phi}$, and also in terms of the angle $\theta$ between $\hat{\psi}$ and $\hat{\phi}$.

**Answer:** Expanding $\rho_\psi \rho_\phi$ as

$$\frac{1}{2}\left(1 + \psi_x X + \psi_y Y + \psi_z Z\right) \frac{1}{2}\left(1 + \phi_x X + \phi_y Y + \phi_z Z\right)$$

yields sixteen terms. Diagonal terms involve $1^2 = X^2 = Y^2 = Z^2 = 1$, and Tr $1 = 2$. Off diagonal terms such as $1X$ are traceless, while terms such as $XY = -YX$ cancel. Hence

$$\text{Tr } \rho_\psi \rho_\phi = \frac{1}{2}\left(1 + \psi_x \phi_x + \psi_y \phi_y + \psi_z \phi_z\right) = \frac{1}{2}\left(1 + \hat{\psi} \cdot \hat{\phi}\right) = \frac{1}{2}\left(1 + \cos\theta\right)$$

(c) Evaluate the expectation value of $\mathbf{S}_{\hat{\psi}}$ in the state $\rho_\phi$.

**Answer:**

$$\begin{aligned} \left\langle \mathbf{S}_{\hat{\psi}} \right\rangle_\phi &= \text{Tr } \rho_\phi \mathbf{S}_{\hat{\psi}} \\ &= \text{Tr}\left[\frac{1}{2}\left(1 + \hat{\phi} \cdot \vec{\sigma}\right) \hat{\psi} \cdot \vec{\sigma}\right] \end{aligned}$$

The calculation is now nearly identical to (b).

$$\left\langle \mathbf{S}_{\hat{\psi}} \right\rangle_{\hat{\phi}} = \phi_x \psi_x + \phi_y \psi_y + \phi_z \psi_z = \hat{\phi} \cdot \hat{\psi} = \cos\theta$$

(d) A pure state density operator evolves in time as

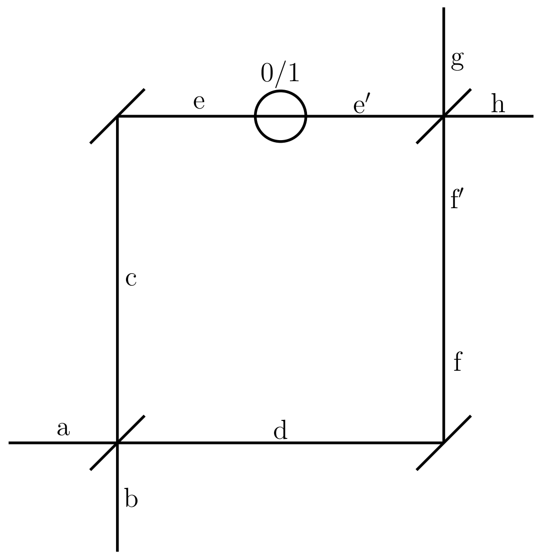
$$\rho(t) = U(t) \rho(0) U^\dagger(t).$$

How would you expect a mixed state density operator to evolve? How would Tr $\rho_\psi \rho_\phi$ evolve?

**Answer:** Since a mixed state density operator is just a linear superposition of pure state operators, and each term in the superposition evolves through conjugation by $U$, then so does the mixed state density operator. A product of density operators evolves as $U(t)\rho_\psi U^\dagger(t)\ U(t)\rho_\phi U^\dagger(t)$. The inner $U^\dagger U = 1$. Cyclic permutation invariance of the trace then shows that Tr $\rho_\psi \rho_\phi$ is time-indepndent.

2. Consider the Mach-Zehnder interferometer shown below. An atom is placed on the arm $e \to e'$. Each letter (including primes) represents a possible state for a photon as it traverses the device. The atom may be in its ground state 0 or its excited state 1. We will model the atom-photon interaction as

$$|e\rangle|0\rangle \to |e'\rangle|1\rangle.$$




(a) What is the dimension of the Hilbert space for the composite photon/atom system?

**Answer:** The dimension is 20, 10 for the photon times 2 for the atom.

(b) Assume a photon enters in state $|a\rangle$ while the atom is in its ground state $|0\rangle$. Determine the final state of the system $|\Psi\rangle$ as the photon exits the interferometer in arm $g$ or $h$.

**Answer:** The unitary evolution is

$$\begin{aligned}
|0\rangle|0\rangle &\to \frac{1}{\sqrt{2}}\left(|c\rangle|0\rangle + |d\rangle|0\rangle\right) \\
&\to \frac{1}{\sqrt{2}}\left(|e\rangle|0\rangle + |f\rangle|0\rangle\right) \\
&\to \frac{1}{\sqrt{2}}\left(|e'\rangle|1\rangle + |f'\rangle|0\rangle\right) \\
&\to \tfrac{1}{2}\left\{(|g\rangle|1\rangle + |h\rangle|1\rangle) + (-|g\rangle|0\rangle + |h\rangle|0\rangle)\right\} \equiv |\Psi\rangle.
\end{aligned}$$

(c) Calculate the conditional probability that the atom is in its excited state given that the photon exits through arm $g$.

**Answer:** We have $P(1|g) = P(1,g)/P(g)$. Working out terms yields

$$P(1,g) = |\langle g1|\Psi\rangle|^2 = 1/4 = P(0,g), \quad P(g) = P(0,g) + P(1,g) = 1/2,$$

hence $P(1|g) = 1/2$.

(d) What additional condition must be imposed on our model of the atom-photon interaction to ensure unitarity?

**Answer:** We require that orthogonal states evolve to orthogonal states, hence we require that

$$|e\rangle|1\rangle \to |e'\rangle|0\rangle.$$

(e) Consider a new atom-photon interaction model in which the atom in its ground state becomes excited, but it absorbs the photon in the process. For clarity, we will denote the state of "no photon" by $|n\rangle$. If the atom is in its excited state it remains excited and the photon passes through unaffected. Specifically,

$$|e\rangle|0\rangle \to |n\rangle|1\rangle, \quad |e\rangle|1\rangle \to |e'\rangle|1\rangle.$$

Repeat your calculation of part (b) for the final state according to this new model.

**Answer:**

$$\begin{aligned}|0\rangle|0\rangle &\to \frac{1}{\sqrt{2}}\left(|n\rangle|1\rangle + |f'\rangle|0\rangle\right)\\ &\to \frac{1}{\sqrt{2}}|n\rangle|1\rangle + \tfrac{1}{2}\left(-|g\rangle|0\rangle + |h\rangle|0\rangle\right) \equiv |\Psi\rangle.\end{aligned}$$

(f) Repeat your calculation of part (c) under the new model. How (if at all) would your result change if the initial state of the atom were unknown?

**Answer:** Now the atom is excited only in the case the photon is absorbed, so it cannot exit in arm $g$, and hence

$$P(1,g) = |\langle g1|\Psi\rangle|^2 = 0.$$

If the atom was initially excited then we recover the case of an empty (no atom) interferometer. In this case, the final state is $|\Psi\rangle = |h\rangle|1\rangle$, and again the photon does not exit through arm $g$.

Since we know that the atom is **not** excited if it exits through arm $g$, regardless of its initial state, then detecting a photon in arm $g$ proves that the atom is in its ground state, regardless of its initial state. This constitutes an *interaction-free measurement* revealing the atom to be in its ground state even though the photon did not touch the atom. This process was first described by Elitzur and Vaidman in 1993.