NAME: SOLUTIONS

33-658 QCQI   Final Exam   Dec. 14, 2023

This exam has three main questions with several parts, many of which can be solved independently of each other. When asked to "calculate", "evaluate", *etc.* please show your work. When asked to "describe or" or "explain", answer concisely in words. When asked "what is" or "write down" there is no need to show your work or justify the answer. To increase the chances of full or partial credit, write neatly and show your answer clearly. You may use the backs of the pages for scratch work.

0. **Extra credit**

(a) Have you completed your FCE, or will you do so? **Yes** (1 point)/No (0 points)

(b) Did you share comments on the course, or will you do so? **Yes** (2 points)/No (0 points)

(c) Did you complete the student learning questionnaire? **Yes** (5 points)/No (0 points)

1. **The CHSH Game**

Alice and Bob play a game in which they jointly compete against a referee, but they cannot communicate with each other after the game has started. The referee randomly chooses a pair of bits $x, y \in \{0, 1\}$ and gives them to Alice and Bob, respectively. Alice announces a value $a \in \{0, 1\}$ and Bob announces a value $b \in \{0, 1\}$. If the referee's values were $(x, y) = (1, 1)$, then Alice and Bob win only if $a \neq b$. For all other (x, y) pairs they win if $a = b$.

(a) The optimal success rate Alice and Bob can achieve using classical bits is 75%. Describe a strategy that achieves this bound.

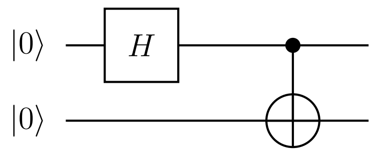
Answer: Alice and Bob always report $a = b = 0$.

(b) Alice and Bob have a strategy to improve their success rate that uses the quantum Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big).$$

Sketch a quantum circuit to create this state.

Answer:



(c) If $x = 0$, Alice will measure her qubit in the z-basis yielding result 0 or 1; if $x = 1$ she will measure in the x-basis yielding a $+$ which she reports as 0 or a $-$ which she reports as 1. If $y = 0$, Bob will measure his qubit in the basis set

$$\{|b_0\rangle = \cos{(\theta)}|0\rangle + \sin{(\theta)}|1\rangle, \quad |b_1\rangle = -\sin{(\theta)}|0\rangle + \cos{(\theta|1\rangle}\},$$

with $\theta = \pi/8$; if $y = 1$, he will use the same basis set but with $\theta = -\pi/8$. Assume the referee chose $x = y = 0$ (it turns out they have the same winning probability for every choice of x and y) and determine the probability that Alice and Bob win.

Answer: If $x = y = 0$, Alice will measure in the z basis and Bob in $\theta = \pi/8$ basis. They both obtain a 0 with probability

$$P(a=0, b=0) = |\big(\langle 0| \otimes \langle b_0|\big)|\Psi\rangle|^2 = \frac{1}{2}\cos^2\left(\frac{\pi}{8}\right)$$

The same holds for $P(a=1, b=1)$, so the win probability is

$$P(\text{win}) = \cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2} + \frac{1}{2\sqrt{2}} = 0.8355...$$

(d) Prove that the conditional probability for Bob to measure the value $b = 0, 1$ (in any basis) given that Alice previously measured $a = 0, 1$ (in any basis) equals the conditional probability for Alice to measure a given that Bob previously measured b. That is, the measurement outcome probabilities commute in time.

Answer: According to Bayes' theorem, $P(b|a) = P(b,a)/P(a) = P(a|b) * P(b)/P(a)$. However, $P(a) = 1/2$ independent of a and basis rotation α, and also $P(b) = 1/2$. To see this, let's calculate $P(a=0)$ by summing over states of b, using the standard basis for b.

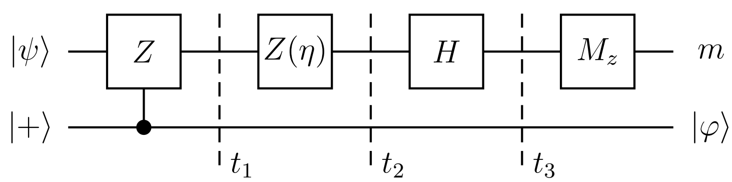
$$\begin{aligned} P(a) &= |(\langle a|_a U_\alpha^\dagger \otimes |0\rangle_b)||\Psi\rangle|^2 + |(\langle a|_a U_\alpha^\dagger \otimes |1\rangle_b|)|\Psi\rangle|^2 \\ &= \frac{1}{2}|\langle a|U_\alpha|0\rangle|^2 + \frac{1}{2}|\langle a|U_\alpha|1\rangle|^2 \\ &= \frac{1}{2}(|\langle a|0\rangle|^2 + |\langle a|1\rangle|^2) = \frac{1}{2} \end{aligned}$$

2. **Measurement-based computation**

In the figure below, the initial single qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, and the phase gate

$$Z(\eta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\eta} \end{pmatrix}$$

represents a rotation by angle η around the z-axis.



(a) Write down the *two*-qubit state $|\Psi(t)\rangle$ at each time t_1, t_2, t_3.

Answer:

$$\begin{aligned} |\Psi(t_1)\rangle &= \frac{1}{\sqrt{2}}\big(|\psi\rangle|0\rangle + (\alpha|0\rangle - \beta|1\rangle)|1\rangle\big) \\ |\Psi(t_2)\rangle &= \frac{1}{\sqrt{2}}\big((\alpha|0\rangle + \beta e^{i\eta}|1\rangle)|0\rangle - \alpha|0\rangle + \beta e^{i\eta}|1\rangle)|1\rangle\big) \\ |\Psi(t_3)\rangle &= \frac{1}{2}\big\{[\alpha(|0\rangle + |1\rangle) + \beta e^{i\eta}(|0\rangle - |1\rangle)]|0\rangle - [\alpha(|0\rangle + |1\rangle) + \beta e^{i\eta}(|0\rangle - |1\rangle)]|1\rangle\big\} \end{aligned}$$

Note that the final state is equivalent to

$$|\Psi(t_3)\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle \otimes HZ(\eta)|\psi\rangle + |1\rangle \otimes ZHZ(\eta)|\psi\rangle\big).$$

This will be useful in parts (b) and (c).

(b) Use the Born rule to express the probability for measurement outcome $m=1$, then evaluate the result.

Answer: According to the Born rule, the probability for $m = 1$ in state $|\Psi(t_3)\rangle$ is

$$P(m = 1) = \langle\Psi(t_3)|\Pi_1|\Psi(t_3)\rangle$$

where the projector onto $|m = 1\rangle$ may be expressed as

$$\Pi_1 = |1\rangle\langle 1| \otimes I.$$

In view of the re-writing of $|\Psi(t_3)\rangle$ above, we read off probability $P(m = 1) = 1/2$ by inspection.

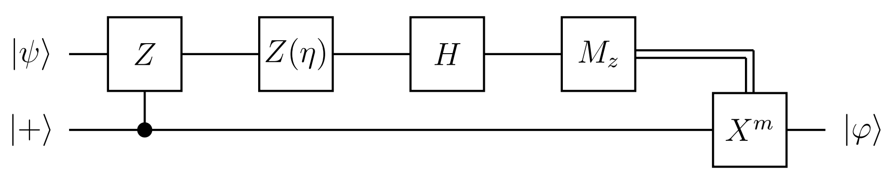
(c) What is the state $|\varphi\rangle$ conditioned on $m = 1$?

Answer: To obtain the conditional state, we apply the projector Π_1 defined above, then divide by $\sqrt{P(m = 1)}$, yielding the conditional state

$$ZHZ(\eta)|\psi\rangle.$$

(d) Show that $HZ^m = X^m H$ $(m = 0, 1)$. Sketch a circuit to perform a *measurement-based computation* of $X(\eta)H|\psi\rangle$ with $X(\eta)$ a rotation by η around $\hat{x}$.

Answer: For $m = 0$, $HZ^0 = X^0 H$ trivially. For $m = 1$ we apply $X = HZH$.



3. **Cavity quantum electrodynamics**

An empty metal box forms a resonant cavity for electromagnetic fields. Let the box contain a single atom that can be in its ground or excited state $\{|g\rangle, |e\rangle\}$. Decay of the excited state creates a single photon $|n = 1\rangle$, while absorption of a photon excites the atom $|1g\rangle \to |0e\rangle$. Let $p^\dagger, p$ be creation and anihilation operators for the photon, and $a^\dagger, a$ be operators that excite or de-excite the atom (all these operators commute). The system evolves according to the Jaynes-Cummings Hamiltonian

$$H = -\left(\frac{\hbar\Omega}{2}\right)(pa^\dagger + p^\dagger a).$$

(a) Verify that H is Hermitian. Express it as a matrix in the basis $\{|pa\rangle\} = \{|0g\rangle, |0e\rangle, |1g\rangle, |1e\rangle\}$.

Answer: The Hermitian conjugate interchanges $pa^\dagger$ with $ap^\dagger$, but a and p apply to different physical subsystems and hence they commute. The Hamiltonian is

$$H = -\left(\frac{\hbar\Omega}{2}\right)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Calculate the quantum state $|\Psi(t)\rangle$ given that $|\Psi(0)\rangle = |0e\rangle$.

Answer: It suffices to restrict our attention to the subspace spanned by $\{|0e\rangle, |1g\rangle\}$. The eigenstates are $|\pm\rangle = (|0e\rangle \pm |1g\rangle)/\sqrt{2}$, with energies $E_\pm = \mp\hbar\Omega/2$. The time evolution operator

$$U(t) = e^{-iHt} = e^{-iE_+t}|+\rangle\langle+| + e^{-iE_-t}|-\rangle\langle-|$$

and

$$|\Psi(t)\rangle = U(t)|0e\rangle = \cos(\Omega t/2)|0e\rangle + i\sin(\Omega t/2)|1g\rangle.$$

(c) At what time $t = T$ does the atom first have a 50% probability of excitation. What is the state $|\Psi(T)\rangle$, and what is the density matrix the atom at that time?

Answer:

$$|\Psi(t = \pi/2\Omega)\rangle = \frac{1}{\sqrt{2}}\big(|0e\rangle - |1g\rangle\big), \quad P(e) = P(g) = \frac{1}{2}$$

The density operator for the atom at time $t = \pi/\Omega$ is

$$\rho_a = \mathrm{Tr}_p|\Psi\rangle\langle\Psi| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

This is a pure state because Tr ρ_a = Tr $\rho_a^2 = 1$.

(d) The atom is removed from the cavity at time T, and then a second atom in its ground state is placed in the cavity for a total time $2T$ (double the time for the first atom). Determine the state $|\Phi\rangle$ of the photon and both atoms at time $2T$. Take as your basis for $|\Phi\rangle$ the set of eight states $|pa_1a_2\rangle$, where $p \in \{0, 1\}$, and a_1 and $a_2 \in g, e$.

Answer: After the first atom is removed from the cavity, the three-particle state is

$$|\Phi(t' = 0)\rangle = |\Psi\rangle \otimes |g\rangle$$

During time $2T$, the first atom is unaffected, but the photon (if present) fully excites the second atom, resulting in

$$|\Phi(t' = 2T)\rangle = \frac{1}{\sqrt{2}}\big(|0eg\rangle - |0ge\rangle\big) \qquad = \frac{1}{\sqrt{2}}|0\rangle \otimes \big(|eg\rangle - |ge\rangle\big).$$

(e) If the first atom is found to be in its ground state, what is the state of the second atom?

Answer: The two atoms are in an entangled Bell state such that their excitations are opposite. If the first atom is in its ground state, the second atom must be excited.