33-658 Quantum Computing and Quantum Information Homework 1

1. The set $\{P_i \ : \ i = 1, \ldots, N\}$ represents the probabilities of N possible events. Prove that the entropy is maximized if all probabilities take the common value $P_i = 1/N$. *Hint:* Use a Lagrange multiplier to enforce the constraint $\sum_i P_i = 1$.

2. Apply Jensen's inequality to prove that the mutual information of two distributions $P(X)$ and $P(Y)$ is necessarily non-negative.

3. The quote "To be, or not to be, that is the question." has 42 characters (not counting the quotation marks), as shown in this table, along with their frequencies f.

character	␣	t	o	e	s	n	i	h	b	,	u	r	q	a	T	.	Total
frequency	9	6	5	4	2	2	2	2	2	2	1	1	1	1	1	1	42

(a) How many bits would the quote occupy in a plain text computer file?

(b) Calculate the entropy in units of bits per character, and the total information content of the full quote.

(c) Create a Shannon code (see https://en.wikipedia.org/wiki/Shannon-Fano_coding) for the quote and determine how many bits are required to store the compressed quote using this code.

4. The unit vector $\mathbf{n}$ with polar angle θ and azimuthal angle φ has Cartesian coordinates $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Let $S_x = (\hbar/2)\sigma_x$, $S_y = (\hbar/2)\sigma_y$, and $S_z = (\hbar/2)\sigma_z$ be the x, y, and z components of the spin operator $\mathbf{S}$, with σ_k the k^{th} Pauli matrix (see SW Eq. 3.35).

(a) Verify that

$$S_{\mathbf{n}} \equiv \mathbf{n} \cdot \mathbf{S} = \frac{\hbar}{2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix}.$$

(b) Evaluate the trace and the determinant of $S_{\mathbf{n}}$.

(c) Show that the vectors

$$|\mathbf{n}^+\rangle = \cos\frac{\theta}{2} e^{-i\varphi/2}|z^+\rangle + \sin\frac{\theta}{2} e^{i\varphi/2}|z^-\rangle$$
$$|\mathbf{n}^-\rangle = -\sin\frac{\theta}{2} e^{-i\varphi/2}|z^+\rangle + \cos\frac{\theta}{2} e^{i\varphi/2}|z^-\rangle.$$

are eigenvectors of $S_{\mathbf{n}}$ and determine their eigenvalues.