33-658 Quantum Computing and Quantum Information Homework 2

1. A two-dimensional Hilbert space has basis elements$\{|0\rangle, |1\rangle\}$.

(i) Express the following operators as matrices in that basis, and show that these matrices are Hermitian.

$$X \equiv |0\rangle\langle 1| + |1\rangle\langle 0|$$
$$Y \equiv -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$
$$Z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|$$

(ii) Evaluate X^2, Y^2, and Z^2.

(iii) Evaluate the commutator $[X, Y]$.

(iv) Define the state $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$. Which of the operators X, Y, and Z have definite values in this state, and which do do not?

2. Consider the density operator for the pure state $|\psi\rangle$, namely $\rho_\psi = |\psi\rangle\langle\psi|$. Show that Tr $\rho_\psi^2 = 1$.

3. An operator P is positive if $\langle\psi|P|\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$. Show that

(i) $|\varphi\rangle\langle\varphi|$ is positive for any $|\varphi\rangle \in \mathcal{H}$.

(ii) If P is positive, and π is a projection operator, then $\pi P \pi$ is also positive.

(iii) If P is positive and $|\psi\rangle$ is normalized, then $\langle\psi|P|\psi\rangle \leq$ Tr P.

4. The exponential of an operator is defined by its power series expansion, *e.g.*

$$\exp \mathbf{A} = 1 + \mathbf{A} + \mathbf{A}^2/2 + \cdots .$$

Let $\hat{\mathbf{n}}$ be a three-dimensional unit vector, and let $\boldsymbol{\sigma}$ be the vector of Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$. Evaluate

$$e^{i(\theta/2)\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}}.$$

5. The only transformation of the classical bits 0 and 1 is logical not, "$\neg$".

(i) What is the most general matrix representation of logical not for a qubit?

(ii) In quantum computing applications we require that $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$ be an eigenstate of $\neg$ with eigenvalue $+1$. What is the form of the matrix X representing $\neg$ subject to that condition?

(iii) What is the most general transformation of a qubit that can be achieved by quantum mechanical time evolution? Give an explicit matrix in the $\{|0\rangle, |1\rangle\}$ basis.