33-658 Quantum Computing and Quantum Information Homework 3

1. Read Schumacher and Westmoreland chapter 5 “Quantum Dynamics”.

   (a) Let $U(t)$ be the time evolution operator for a time-independent Hamiltonian H so that any quantum state $|\psi\rangle$ evolves as $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. Show that $U(t)$ itself solves the Schroedinger equation

   $$i\hbar\frac{d}{dt}U(t) = HU(t).$$

   (b) Let the basis set $\{|e_i\rangle\}$ be a complete set of eigenstates of the Hamiltonian, namely $H|e_i\rangle = E_i|e_i\rangle$. Write down the spectral decomposition of the time evolution operator using projectors onto this basis.

   (c) Consider a spin 1/2 particle in a magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$ pointing along the x axis. The magnetic moment $\mathbf{m} = \gamma\mathbf{S}$, and the initial state $|\psi(0)\rangle = |z^+\rangle$. Express the state at time t, using the $|z^\pm\rangle$ basis.

   (d) Evaluate the probability $P(t)$ that $S_y = +\hbar/2$ as a function of time. *Note:* $|y^+\rangle = (|z^+\rangle + i|z^-\rangle)/\sqrt{2}$.

2. A spin 1 particle has states $\{|z^+\rangle, |z^0\rangle, |z^-\rangle\}$, with values of $S_z = +\hbar, 0, -\hbar$, respectively. As given in Shumacher and Westmoreland Eq. (3.16), the corresponding eigenstates of S_x are

   $$\begin{aligned}
   |x^+\rangle &= \frac{1}{2}(|z^+\rangle + \sqrt{2}|z^0\rangle + |z^-\rangle) \\
   |x^0\rangle &= \frac{1}{\sqrt{2}}(|z^+\rangle - |z^-\rangle) \\
   |x^-\rangle &= \frac{1}{2}(|z^+\rangle - \sqrt{2}|z^0\rangle + |z^-\rangle)
   \end{aligned}$$

   (a) Express the operator S_x as a 3×3 matrix in the z basis. *Hint:* Use the projectors onto S_x eigenstates expressed in the z basis.

   (b) Express the transformation from the z basis to the x basis as a 3×3 unitary matrix, U.

   (c) Apply the unitary transformation U to convert your S_x matrix in the z basis (as obtained in part (a)) into the x basis.

3. Answer Schumacher and Westmoreland problem 4.8, “Quantum Money”. Note that part (b) is ambiguous. You should work out the probability that the counterfeiter has the correct basis, regardless of whether they are certain. Also note that there is an obvious strategy with a certain

probability of success, and a more subtle strategy with a higher probability. See if you can work out both.

4. Set up an account at https://quantum-computing.ibm.com/, run the following two circuits on one of their quantum computers, and show your measurement outcomes. Discuss the quantum and classical states before and after each operation, including probabilities and correlations, assuming the runs were fault-free.