1. Dense coding.

The Bell states defined by

$$|B_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |B_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$

$$|B_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad |B_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

form an orthonormal basis of fully entangled states on the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$ of two two-dimensional Hilbert spaces, where each space has an orthonormal basis $\{|0\rangle, |1\rangle\}$. Note that

$$|B_{xy}\rangle = \frac{1}{\sqrt{2}} (|0y\rangle + (-1)^x |1\bar{y}\rangle)$$

with $\bar{y} = \neg y$.

(a) Find unitary operations of the form $U_a \otimes I_b$ that map $|B_{00}\rangle$ to $|B_{01}\rangle$, $|B_{10}\rangle$, and $|B_{11}\rangle$. Do you recognize the $U_a$ operators?

(b) Alice and Bob share a Bell state $B_{xy}$ but they do not know the values of $x$ or $y$. Two bits of information are required to specify $xy$. How many bits of information concerning $xy$ are obtained if Alice measures the value of her qubit $a$? How many bits of information are obtained if Bob also measures his qubit $b$ and shares the result with Alice?

(c) Now let Alice and Bob each have two bits, $|a\rangle$ and $|\bar{a}\rangle$, and $|b\rangle$ and $|c\rangle$, respectively, as shown in the figure. The state at time 2 is a product state, $|\Psi_2\rangle = |a\rangle|\bar{a}\rangle|B_{00}\rangle$. Bob passes bit $b$ to Alice, who acts on it between times 3 and 5, then returns it to Bob. What is the state at time 6?

(d) The figure claims that the final state at time 8 is $|a\bar{a}a\rangle$. Explain why this is true.

(e) The final state at time 8 contains two copies each of bits $a$ and $\bar{a}$. Why does this not violate the no-cloning theorem?
2. Schumacher & Westmoreland problem #7.3 “Preparation machines”

3. Schumacher & Westmoreland problem #7.5 Teleportation.