1. (a) Show that, given a density matrix $\rho$, its Bloch vector can be determined via the relation

$$a_i = Tr(\rho \sigma_i).$$  \hspace{1cm} (1)

(b) A Qbit goes through a noisy channel whose Kraus operators are given by

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}.$$ 

(c) Calculate the output $\rho'$ in terms of the input $\rho$ which has matrix elements $\rho_{ij}$.

(d) Assuming that the initial density matrix has vanishing elements except for $\rho_{11}$. Determine the final state Bloch vector $\vec{a}$ as a function of $\rho_{11}$.

(e) Suppose that the initial state is $|1\rangle\langle 1|$. The circuit is applied $n$ times over an interval $n\Delta t = T$, find the decoherence rate ($\Gamma$) defined by $\rho_{11}(t) = \rho_{11}(0)e^{-\Gamma T}$.

(f) Calculate the information lost in the channel.

(g) Calculate the Fidelity of this noisy channel.

(h) The operator $R_y(\pm 2\theta)$ is defined as a rotation around the y-axis $R = e^{\pm i\sigma_y\theta}$. Calculate the matrix form of this operator.

(i) Consider the quantum circuit shown in the figure, it implements this quantum channel. Where the gates labeled $\pm \theta/2$ are rotations matrices $R_y(\mp \theta)$. We would like to write this circuit as a four by four unitary matrix. To do so work in the four by four basis ($|00\rangle, |10\rangle, |01\rangle, |11\rangle$). The first/second bit we will call the target/control bits ($|TC\rangle$). Write the circuit as a product of four four by four matrices. Write out each matrix to get maximal credit.
FIG. 1. Quantum circuit which implements the operation associated with the Krauss operators in problem 1.

(j) Multiply out the matrices (feel free to use Mathematica or any other package you wish to reduce the chances of error). Make sure to get the order of the matrices correct. Check your result by making sure it's unitary.) Hint: Recall that given a two by two matrix $A$,

$$A \otimes 1 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

(k) Suppose the initial state is $|01\rangle$. Operate on your state with the unitary 4 by 4 representing the quantum circuit to determine the final state of the system.

(l) Compare your results to your results in part (d) to determine $p$ in terms of $\theta$.

2. A Qbit is in the state $\alpha |0\rangle + \beta |1\rangle$. It interacts with an environment (pointer) Qbit initially in the state $|0\rangle_E$. The final state is $\alpha |0\rangle |0\rangle_E + \beta |1\rangle |1\rangle_E$. You want to determine if the Qbits state is in one of two states $\alpha |0\rangle \pm i\beta |1\rangle$.

(a) Where measurement should you perform on the pointer?

(b) What are the associated measurement operators?

(c) What are the POVM operators $E_{\pm}$.

3. A Qbit $\alpha |0\rangle + \beta |1\rangle$ passes through a noisy channel the effect of which is to generate a rotation on the bit via the unitary $U = e^{i\hbar \sigma_\epsilon}$. 

(a) Show that this corresponds to a general error composed of a weighted bit flip and
phase flip. Determine the state after passing through the channel in terms of the components of $\hat{n}$, $\epsilon$, $\alpha$ and $\beta$.

(b) The Shor 9-bit error correction code is used in this channel. Determine the final state. Does the code return the system to the original state at order $\epsilon$? Note just assume the code does what its suppose to do. i.e. you dont have to simulate the code.