1. The set \( \{P_i : i = 1, \ldots, N\} \) represents the probabilities of \( N \) possible events. Prove that the entropy is maximized if all probabilities take the common value \( P_i = 1/N \). \textit{Hint:} Use a Lagrange multiplier to enforce the constraint \( \sum_i P_i = 1 \).

2. Apply Jensen’s inequality to prove that the mutual information of two distributions \( P(X) \) and \( P(Y) \) is necessarily non-negative.

3. The quote “To be, or not to be, that is the question.” has 42 characters (not counting the quotation marks), as shown in this table, along with their frequencies \( f \).

<table>
<thead>
<tr>
<th>character</th>
<th>~ t o e s n i h b , u r q a T .</th>
<th>Total frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>9 6 5 4 2 2 2 2 2 1 1 1 1 1 1 1</td>
<td>42</td>
</tr>
</tbody>
</table>

(a) How many bits would the quote occupy in a plain text computer file?
(b) Calculate the entropy in units of bits per character, and the total information content of the full quote.
(c) Create a Shannon code (see https://en.wikipedia.org/wiki/Shannon-Fano_coding) for the quote and determine how many bits are required to store the compressed quote using this code.

4. The unit vector \( \mathbf{n} \) with polar angle \( \theta \) and azimuthal angle \( \varphi \) has Cartesian coordinates \((\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\). Let \( S_x = (\hbar/2)\sigma_x \), \( S_y = (\hbar/2)\sigma_y \), and \( S_z = (\hbar/2)\sigma_z \) be the \( x \), \( y \), and \( z \) components of the spin operator \( \mathbf{S} \), with \( \sigma_k \) the \( k \)th Pauli matrix (see SW Eq. 3.35).
   a) Verify that \( S_n \equiv \mathbf{n} \cdot \mathbf{S} = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & -\cos \theta \end{pmatrix} \).
   
   (b) Evaluate the trace and the determinant of \( S_n \).
   
   c) Show that the vectors
   \[
   |n^+\rangle = \cos \frac{\theta}{2} e^{-i\varphi/2} |z^+\rangle + \sin \frac{\theta}{2} e^{i\varphi/2} |z^-\rangle \\
   |n^-\rangle = -\sin \frac{\theta}{2} e^{-i\varphi/2} |z^+\rangle + \cos \frac{\theta}{2} e^{i\varphi/2} |z^-\rangle 
   \]
   are eigenvectors of \( S_n \) and determine their eigenvalues.