1. A two-dimensional Hilbert space has basis elements \(|0\rangle, |1\rangle\).

(i) Express the following operators as matrices in that basis, and show that these matrices are Hermitian.

\[
X \equiv |0\rangle\langle 1| + |1\rangle\langle 0|
\]
\[
Y \equiv -i|0\rangle\langle 1| + i|1\rangle\langle 0|
\]
\[
Z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|
\]

(ii) Evaluate \(X^2\), \(Y^2\), and \(Z^2\).

(iii) Evaluate the commutator \([X, Y]\).

(iv) Define the state \(|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}\). Which of the operators \(X\), \(Y\), and \(Z\) have definite values in this state, and which do not?

2. Consider the density operator for the pure state \(|\psi\rangle\), namely \(\rho_\psi = |\psi\rangle\langle \psi|\). Show that \(\text{Tr} \rho_\psi^2 = 1\).

3. An operator \(P\) is positive if \(\langle \psi | P | \psi \rangle \geq 0\) for all \(|\psi\rangle \in \mathcal{H}\). Show that

(i) \(|\varphi\rangle \langle \varphi|\) is positive for any \(|\varphi\rangle \in \mathcal{H}\).

(ii) If \(P\) is positive, and \(\pi\) is a projection operator, then \(\pi P \pi\) is also positive.

(iii) If \(P\) is positive and \(|\psi\rangle\) is normalized, then \(\langle \psi | P | \psi \rangle \leq \text{Tr} P\).

4. The exponential of an operator is defined by its power series expansion, e.g.

\[
\exp A = 1 + A + A^2/2 + \cdots.
\]

Let \(\hat{n}\) be a three-dimensional unit vector, and let \(\sigma\) be the vector of Pauli matrices \((\sigma_x, \sigma_y, \sigma_z)\). Evaluate

\[
\exp (i\theta/2) \hat{n} \cdot \sigma.
\]

5. The only transformation of the classical bits 0 and 1 is logical not, “\(\neg\)”.

(i) How would you represent logical not for a qubit? Is this representation unique?

(ii) What is the result of quantum \(\neg\) acting on the state \(|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\)?

(iii) What is the most general transformation of a qubit? Give an explicit matrix in the \(|0\rangle, |1\rangle\) basis.