1. Dense coding.

The Bell states defined by

\[ |B_{00}⟩ = (|00⟩ + |11⟩)/\sqrt{2}, \quad |B_{01}⟩ = (|01⟩ + |10⟩)/\sqrt{2}, \]
\[ |B_{10}⟩ = (|00⟩ - |11⟩)/\sqrt{2}, \quad |B_{11}⟩ = (|01⟩ - |10⟩)/\sqrt{2} \]

form an orthonormal basis of fully entangled states on the tensor product \( \mathcal{H}_a \otimes \mathcal{H}_b \) of two two-dimensional Hilbert spaces, where each space has an orthonormal basis \{ |0⟩, |1⟩ \}. Note that

\[ |B_{xy}⟩ = \frac{1}{\sqrt{2}} (|0y⟩ + (-1)^x|1\bar{y}⟩) \]

with \( \bar{y} = \neg y \).

(a) Find unitary operations of the form \( U_a \otimes I_b \) that map \( |B_{00}⟩ \) to \( |B_{01}⟩, |B_{10}⟩, \) and \( |B_{11}⟩ \). Do you recognize the \( U_a \) operators?

(b) Alice and Bob share a Bell state \( B_{xy} \) but they do not know the values of \( x \) or \( y \). Two bits of information are required to specify \( xy \). How many bits of information concerning \( xy \) are obtained if Alice measures the value of her qubit \( a \)? How many bits of information are obtained if Bob also measures his qubit \( b \) and shares the result with Alice?

(c) Now let Alice and Bob each have two bits, \( |a⟩ \) and \( |\bar{a}⟩ \), and \( |b⟩ \) and \( |c⟩ \), respectively, as shown in the figure. The state at time 2 is a product state, \( |Ψ_2⟩ = |a⟩|\bar{a}⟩|B_{00}⟩ \). Bob passes bit \( b \) to Alice, who acts on it between times 3 and 5, then returns it to Bob. What is the state at time 6?

(d) The figure claims that the final state at time 8 is \( |a\bar{a}a\bar{a}⟩ \). Explain why this is true.

(e) The final state at time 8 contains two copies each of bits \( a \) and \( \bar{a} \). Why does this not violate the no-cloning theorem?
2. Schumacher & Westmoreland problem #7.3 “Preparation machines”

3. Schumacher & Westmoreland problem #7.5 Teleportation.