Zero point displacements of the harmonic chain

(a) Show that the expectation values of kinetic and potential energy of the harmonic oscillator obey the virial relation for quadratic potentials, $\langle V\rangle = \langle K\rangle$ when in an energy eigenstate. Use this result to calculate the mean square displacement $\langle X^2\rangle$ in the ground state. Show, further, that the time average expectation value $\overline{\langle V\rangle} = \overline{\langle K\rangle}$ regardless of the quantum state.

(b) Consider the infinite periodic chain of coupled oscillators discussed by Cohen-Tannoudji (complement J_V). Express the position of the j^{th} oscillator, X_j in terms of the normal mode coordinates $\Xi(k)$, with $k \in (-\pi/l, \pi/l)$, where l is the period of the chain.

(c) Evaluate the expectation value $\langle \Xi(k)\Xi^\dagger(k')\rangle$ in the ground state.

(d) Let the potential U vanish but keep V nonzero (i.e. $\omega = 0$ but $\omega_1 \neq 0$ in Cohen-Tannoudji's notation). Show that the mean square displacement $\langle X_j^2\rangle$ of each mass j diverges in the ground state, but the mean square separation of neighboring masses $\langle (X_{j+1} - X_j)^2\rangle$ remains finite.