The following two problems both require unitary transformations operating on qubits ("quantum gates"). You may wish to recall the following:

\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix},
\]

\[
cZ = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix},
\]

\[
XOR = cX = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

1. (Dense coding, 40 points)

The figure below illustrates a device for "dense coding". Here the initial Hadamard and XOR gates prepare an entangled Bell state \( |B_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \). Then bit \( b \) of this entangled state is carried without disturbing it from Bob’s lab to Alice’s lab at a remote location. At time \( t_3 \), Alice creates a pair of qubits, \( |a\rangle \) and \( |\bar{a}\rangle \) and operates on the three qubit system \( |a\bar{a}b\rangle \) as indicated. The resulting bit \( b \) is carried back to Bob’s lab and operated on as shown. For each of the four combinations of \( a, \bar{a} = 0, 1 \), identical values are output in Bob’s lab. It appears that a single qubit (\( b \)) has carried two bits of information (\( a \) and \( \bar{a} \)).
(a) We wish to examine the states at time $t_5$, after Alice has prepared the bit $b$ she will give to Bob. Here is an example evolution with $|a\bar{a}\rangle = |11\rangle$:

$$|\Psi_3\rangle = |11\rangle \otimes |B_0\rangle \rightarrow |\Psi_4\rangle = |11\rangle \otimes (|10\rangle + |01\rangle) / \sqrt{2}$$

$$\rightarrow |\Psi_5\rangle = |11\rangle \otimes (-|10\rangle + |01\rangle) / \sqrt{2}.$$  

Work out $|\Psi_5\rangle$ for the other three cases, $|a\bar{a}\rangle = |00\rangle, |01\rangle$ and $|10\rangle$. 


(b) Evaluate the probabilities $\Pr(b = 0|\Psi_5)$ and $\Pr(b = 1|\Psi_5)$ for all four combinations of $a, \bar{a}$.

How does qubit $b$ actually carry information?

(c) If Alice put the general state $\psi = \alpha|0\rangle + \beta|1\rangle$ into channel $a$, would $\psi$ emerge in output $a$ of Bob’s lab? Give a yes/no answer with a one sentence justification.
2. (Measure qubit in W-basis, 60 points)

(a) Assume that a qubit has been sent in the basis \{|w^+\rangle, |w^-\rangle\}. Construct an operator \(W\) that transforms this into the basis of “pointer states” \{\ket{0}, \ket{1}\}. That is, \(W|w^+\rangle = |0\rangle\) and \(W|w^-\rangle = |1\rangle\). Prove that \(W\) is a unitary transformation.

(b) We wish to measure the qubit using the quantum circuit drawn in the figure below. Determine the final state \(\Psi_2\) if the input qubit \(|a\rangle = |w^+\rangle\) and also if \(|a\rangle = |w^-\rangle\).
(c) Assuming an initial state

$$|\Psi_0\rangle = (\alpha |w^+\rangle + \beta |w^-\rangle) \otimes |0\rangle,$$

construct a complete and consistent family of three-time histories (at \(t_0, t_1\) and \(t_2\)) that can correlate measurement outcomes \([0]_b\) and \([1]_b\) at time \(t_2\) with properties of the qubit \(a\) at time \(t_1\). Show that the history family is consistent.
(d) Evaluate the conditional probabilities $\Pr([a]_1|[b]_2)$ where $a = w^+, w^-$ and $b = 0, 1$. 