1. Hydrogen atom in magnetic field (adapted from Cohen-Tannoudji complement D_{VI}).

Consider a hydrogen atom in a magnetic field \(B\). Neglecting spin, the Hamiltonian is

\[
H = \frac{1}{2m_e} |p - qA(R)|^2 + V(R)
\]

where the vector potential in the Coulomb gauge is \(A = -\frac{1}{2} R \times B\) and the Coulomb potential is \(V(R) = -\frac{q^2}{R}\).

(a) Expand \(H\) in powers of \(B\) as \(H_0 + H_1 + \cdots\), keeping terms only through first order in \(B\).

Express the first order contribution, \(H_1\), in terms of the angular momentum operator \(L\), and the Bohr magneton \(\mu_B = q\hbar/2m_e\). You may assume \(B = B\hat{z}\). 
(b) In terms of the $H_0$ eigenstates $\{|nlm\}\rangle$, with eigenvalue $-E_I/n^2$ ($E_I = 13.6$ eV), determine the eigenstates and corresponding eigenvalues of $H_0 + H_1$.

(c) List the components $\alpha \in (x, y, z)$ of $D = qR$ and the azimuthal angular momenta $m = \pm 1, 0$ for which the matrix elements of $\langle 100 | D_\alpha | 21m \rangle$ vanish. You should briefly justify your answers.
(d) Recalling that oscillating electric dipoles radiate, sketch a plausible Lyman radiation spectrum for \( n = 2 \rightarrow n' = 1 \) transitions in the presence of a magnetic field and label the frequencies of any key features. What happens to this spectrum in the special case of photon propagation parallel to \( \mathbf{B} \)?
2. Harmonic oscillator in the momentum representation (adapted from complement $D_V$ of Cohen-Tannoudji).

The time-independent Schrödinger equation of a harmonic oscillator expressed as a differential equation in the position representation is

$$\left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2 \right\} \varphi(x) = E\varphi(x).$$

Its eigenfunctions are $\varphi_n(x)$ with energy $E_n = (n + \frac{1}{2})\hbar\omega$.

(a) Write down the Schrödinger equation as a differential equation in the momentum representation governing the Fourier transformed wave function

$$\tilde{\varphi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx e^{-ipx/\hbar} \varphi(x).$$
(b). Comparing the Schrödinger equations in the $x$ and $p$ representations, find the general relationship between $\tilde{\varphi}_n(p)$ and $\varphi_n(x)$. In other words, you must express $\tilde{\varphi}_n(p)$ in terms of $\varphi_n(x)$, uniquely up to a phase factor $e^{i\theta_n}$, without using a Fourier transform. Explain your reasoning in a sentence or two.

(c). Recalling that

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \quad \text{and} \quad P = i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a),$$

evaluate the expectation values

$$\langle (\Delta X_n)^2 \rangle = \langle \varphi_n | X^2 | \varphi_n \rangle \quad \text{and} \quad \langle (\Delta P_n)^2 \rangle = \langle \tilde{\varphi}_n | P^2 | \tilde{\varphi}_n \rangle.$$

From this result, determine the product $\Delta X_n \Delta P_n$. Compare this result with the Heisenberg uncertainty principle and comment.
3. 1-D potential

Consider a particle propagating in the one-dimensional potential $V(x)$ of arbitrary shape illustrated below that vanishes outside $x \in (-a, a)$. The time-independent Schrodinger equation has solutions of the form

$$
\varphi(x) = \begin{cases} 
    Ae^{ikx} + Be^{-ikx} & (x < -a) \\
    Ce^{ikx} + De^{-ikx} & (a < x)
\end{cases}
$$

where the coefficients $A - D$ are related by the matrix equation

$$
\begin{pmatrix} 
    C \\
    D 
\end{pmatrix} = M(k) \begin{pmatrix} 
    A \\
    B 
\end{pmatrix}, \quad M(k) = \begin{pmatrix} 
    E & F \\
    G & H 
\end{pmatrix}
$$

and the matrix elements $E - H$ are functions of $k = \sqrt{2mE/\hbar^2}$ that depend in some complicated fashion on $V(x)$.

\[ \text{\hspace{10cm}} \]

a) The identities $H = E^*$ and $F = G^*$ hold. Briefly explain why.
b) What relation among the matrix elements expresses the condition of current conservation?

c) Determine $T_L$, the transmission coefficient for particles incident from the left, crossing the potential, and escaping to the right. Compare with the value of $T_R$, the transmission coefficient for particles traveling right to left. How does the asymmetry of $V(x)$ affect your conclusions?