NAME: SOLUTIONS

33-755 Quantum I   Midterm Exam #1 (AM)   Friday, Sept 27, 2019

1. Consider a pair of spin-1/2 particles, a and b with the tensor product Hilbert space $\tilde{\mathcal{H}} = \mathcal{H}_a \otimes \mathcal{H}_b$. The Pauli matrices for a single spin-1/2 particle,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

represent the spin components S_x, S_y, and S_z in the basis $\{|z^+\rangle, |z^-\rangle\}$ (in that order). Note we have set $\hbar/2 = 1$, and recall the identities $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$ and $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k$.

(a) Write down the matrix acting on the Hilbert space $\tilde{\mathcal{H}}$ that represents the spin component S_{ax} of particle a in the x-direction. Specify the basis that you have chosen, including the order in which you take the basis elements.

Answer: Take the basis set $\{|z_a^+ z_b^+\rangle, |z_a^- z_b^+\rangle, |z_a^+ z_b^-\rangle, |z_a^- z_b^-\rangle$. We can write

$$S_{ax} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(b) Without calculation, evaluate the operator $D \equiv S_{ay}S_{bx} - S_{bx}S_{ay}$. Justify your assertion in one or two brief sentences.

Answer: $D = 0$ because $S_{ay} = \sigma_y^{(a)} \otimes I_b$ acts on different components of the Hilbert space than $S_{bx} = I_a \otimes \sigma_x^{(b)}$. Within the a and b components separately, σ comutes with I.

(c) List all the eigenvalues (including multiplicities) of S_{ax}. Note that S_{bx}, S_{ay}, and S_{by} share the same list as S_{ax}.

> **Answer:** The eigenvalues are $+1$ and -1, each with multiplicity 2.

(d) Let $|\psi\rangle$ be a simultaneous eigenvector of S_{ax} (with eigenvalue a_x) and S_{bx} (with eigenvalue b_x). Evaluate $S_{ax}S_{bx}|\psi\rangle$.

> **Answer:** $S_{ax}S_{bx}|\psi\rangle = S_{ax}b_x|\psi\rangle = a_x b_x|\psi\rangle$.

(e) Consider the following 3×3 array of operators

$$\begin{array}{ccc} S_{ax} & S_{bx} & S_{ax}S_{bx} \\ S_{by} & S_{ay} & S_{ay}S_{by} \\ S_{ax}S_{by} & S_{bx}S_{ay} & S_{az}S_{bz} \end{array}$$

The operators in any given row, and also their product ($e.g.$ $P_x \equiv S_{ax}\ S_{bx}\ (S_{ax}S_{bx})$, *etc.*), commute with each other, and hence may take definite values simultaneously. For each of the three rows, express the product of its operators ($e.g.$ P_x) as an operator on $\tilde{\mathcal{H}}$ (or as a matrix, if you prefer, but it is faster and more efficient to work with operators) and list its eigenvalues (including multiplicities).

> **Answer:** Since S_a components commute with S_b components, and $S_{ax}^2 = I_a \otimes I_b$, then $P_x = \tilde{I}$. The same holds for the second row, P_y, and the third row, P_z. The identity operators in $\tilde{\mathcal{H}}$ have eigenvalue $+1$ with multiplicity 4.

(f) Repeat (e) for each of the three columns. Be careful because the third column is a bit different than the first two.

Answer: The first two columns also multiply to $\tilde{I}$, with eigenvalue $+1$ of multiplicity 4. The final column multiplies to $-\tilde{I}$ because $\sigma_x \sigma_y = i\sigma_z$ within both the a and b components of $\tilde{\mathcal{H}}$, and $i^2 = -1$. The eigenvalue -1 has multiplicity 4.

(g) Make a 3×3 array of possible values of each of the operators. You can randomly choose from among their eigenvalues. Can you assign values consistently with the row and column products you discovered in parts (e) and (f)? If not, provide a brief explanation why not.

Answer: Each entry is $+1$ or -1. Here is one attempt that has the proper products ($+1$) in all three rows, and the first two columns. However the product of the third column is $+1$, while the eigenvalue of $-\tilde{I}$ must be -1. In fact, there is no possible consistent assignment because the product of row products $\tilde{I}^3$ differs from the product of column products $\tilde{I}^2\,(-\tilde{I}) = -\tilde{I}^3$. This shows that we cannot simultaneously assign values of S_x, S_y, and S_z to each of spins a and b. We already knew this because of the incompatibility of the spin components for a single spin. This finding is known as Mermin's two-spin paradox.

$$
\begin{array}{ccc|c}
-1 & +1 & -1 & +1 \\
+1 & +1 & +1 & +1 \\
-1 & +1 & -1 & +1 \\
\hline
+1 & +1 & \mathbf{+1\,!} & ?
\end{array}
$$