NAME: ______________________________________

33-755 Quantum I   Midterm Exam #1 (AM)   Friday, Sept 27, 2019

1. Consider a pair of spin-1/2 particles, a and b with the tensor product Hilbert space $\tilde{\mathcal{H}} = \mathcal{H}_a \otimes \mathcal{H}_b$. The Pauli matrices for a single spin-1/2 particle,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

represent the spin components S_x, S_y, and S_z in the basis $\{|z^+\rangle, |z^-\rangle\}$ (in that order). Note we have set $\hbar/2 = 1$, and recall the identities $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$ and $\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k$.

(a) Write down the matrix acting on the Hilbert space $\tilde{\mathcal{H}}$ that represents the spin component S_{ax} of particle a in the x-direction. Specify the basis that you have chosen, including the order in which you take the basis elements.

(b) Without calculation, evaluate the operator $D \equiv S_{ay}S_{bx} - S_{bx}S_{ay}$. Justify your assertion in one or two brief sentences.

(c) List all the eigenvalues (including multiplicities) of S_{ax}. Note that S_{bx}, S_{ay}, and S_{by} share the same list as S_{ax}.

(d) Let $|\psi\rangle$ be a simultaneous eigenvector of S_{ax} (with eigenvalue a_x) and S_{bx} (with eigenvalue b_x). Evaluate $S_{ax}S_{bx}|\psi\rangle$.

(e) Consider the following 3×3 array of operators

$$\begin{array}{ccc} S_{ax} & S_{bx} & S_{ax}S_{bx} \\ S_{by} & S_{ay} & S_{ay}S_{by} \\ S_{ax}S_{by} & S_{bx}S_{ay} & S_{az}S_{bz} \end{array}$$

The operators in any given row, and also their product (*e.g.* $P_x \equiv S_{ax}\ S_{bx}\ (S_{ax}S_{bx})$, *etc.*), commute with each other, and hence may take definite values simultaneously. For each of the three rows, express the product of its operators (*e.g.* P_x) as an operator on $\tilde{\mathcal{H}}$ (or as a matrix, if you prefer, but it is faster and more efficient to work with operators) and list its eigenvalues (including multiplicities).

(f) Repeat (e) for each of the three columns. Be careful because the third column is a bit different than the first two.

(g) Make a 3×3 array of possible values of each of the operators. You can randomly choose from among their eigenvalues. Can you assign values consistently with the row and column products you discovered in parts (e) and (f)? If not, provide a brief explanation why not.