NAME: SOLUTIONS

33-755 Quantum I Midterm Exam #1 (PM) Friday, Sept 27, 2019

1. Two spin-1/2's (adapted from Cohen-Tannoudji #IV.6 and IV.7)

Consider the state $|\psi_0\rangle = \alpha|++\rangle + \beta|+-\rangle + \gamma|-+\rangle + \delta|--\rangle$, where $|\pm\rangle$ are the up and down states along the $\hat{z}$ axis for each spin a and b.

(a) What condition must the coefficients satisfy so that the ket-vector is normalized?

Answer: We require that $\langle\psi_0|\psi_0\rangle = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.

(b) Under what condition is $|\psi_0\rangle$ a product state? Briefly explain how you know.

Answer: The most general product state can be written

$$(\Psi_{a+}|+\rangle + \Psi_{a-}|-\rangle) \otimes (\Psi_{b+}|+\rangle + \Psi_{b-}|-\rangle) =$$

$$\Psi_{a+}\Psi_{b+}|++\rangle + \Psi_{a+}\Psi_{b-}|+-\rangle\Psi_{a-}\Psi_{b+}|-+\rangle + \Psi_{a-}\Psi_{b-}|--\rangle$$

Comparing coefficients with $|\psi_0\rangle$ we see the condition is $\alpha\delta = \beta\gamma$.

(c) Express the property "$S_{bx} = +\hbar/2$" as a matrix in the basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$, then use this matrix to find the corresponding probability.

Answer: For a single spin, $|x+\rangle = (|+\rangle + |-\rangle)/\sqrt{2}$ so that

$$[x+] = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Taking the tensor product of I on spin a with $[x+]$ on spin b yields

$$[x_b+] = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now we can evaluate the probability as

$$\text{Pr } = |[x_b+]|\psi_0\rangle|^2 = |\langle\psi_0|[x_b+]|\psi_0\rangle = \alpha^*(\alpha+\beta) + \beta*(\alpha+\beta) + \gamma^*(\gamma+\delta) + \delta^*(\gamma+\delta)$$

(d) Recall (from this morning!) that

$$S_{bx} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and use it to calculate the mean value $\langle S_{bx} \rangle$.

Answer:

$$\langle S_{bx} \rangle = \langle \psi_0 | S_{bx} | \psi_0 \rangle = \alpha^* \beta + \beta^* \alpha + \gamma^* \delta + \delta^* \gamma$$

(e) The state evolves from $|\psi_0\rangle$ for a time t under the Hamiltonian $H = \omega_1 S_{az} + \omega_2 S_{az}$. Express $|\psi(t)\rangle$ in the usual $|\pm\pm\rangle$ basis.

Answer: Since the basis states are eigenstates of the Hamiltonian, $H|\pm\pm\rangle = (\hbar/2)(\pm\omega_1 \pm \omega_2)|\pm\pm\rangle$, we can write

$$\begin{aligned} |\psi(t)\rangle =& \alpha e^{-i(\omega_1+\omega_2)t/2}|++\rangle + \beta e^{-i(\omega_1-\omega_2)t/2}|+-\rangle \\ &+ \gamma e^{-i(-\omega_1+\omega_2)t/2}|-+\rangle + \delta e^{-i(-\omega_1-\omega_2)t/2}|--\rangle \end{aligned}$$

(f) Letting $\alpha = \beta = \gamma = \delta = 1/2$, determine the mean value $\langle S_{bx} \rangle(t)$.

Answer: Adapting our result from (d), substitutig the coefficients from (e), we have $\langle S_{bx} \rangle = \cos(\omega_2 t)$.