NAME: SOLUTIONS

33-755 Quantum I   Midterm Exam #2   Monday, Oct. 28, 2019

1. 3 Box Paradox

Consider a toy model with three locations for a particle. Call these three orthogonal states $|A\rangle$, $|B\rangle$ and $|C\rangle$. Take the time evolution to be trivial, $T = I$; if the particle is in one of the boxes, it stays there. We consider histories starting in the symmetric initial state

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle)$$

at time t_0 and ending in the final state

$$|F\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle),$$

or its complement, at time t_2. The family of histories $\mathcal{A}$ defined by

$$[\psi_0] \odot \begin{cases} [A] \odot [F] & (Y^{(1)}) \\ [A] \odot (I - [F]) & (Y^{(2)}) \\ (I - [A]) \odot [F] & (Y^{(3)}) \\ (I - [A]) \odot (I - [F]) & (Y^{(4)}) \end{cases}$$

together with

$$Z \equiv (I - [\psi_0]) \odot I \odot I.$$

is complete, compatible and consistent.

(a) Evaluate the chain kets $|Y^{(1)}\rangle, \ldots, |Y^{(4)}\rangle$ and the probabilities $\Pr(Y^{(1)}), \ldots, \Pr(Y^{(4)})$.

Answer: The chain kets are

$$\begin{aligned}
|Y^{(1)}\rangle &= [F][A]|\psi_0\rangle \;=\; [F]\frac{1}{\sqrt{3}}|A\rangle \;=\; \frac{1}{3}|F\rangle, \\
|Y^{(2)}\rangle &= (I-[F])[A]|\psi_0\rangle \;=\; (I-[F])\frac{1}{\sqrt{3}}|A\rangle \;=\; \frac{1}{\sqrt{3}}|A\rangle - \frac{1}{3}|F\rangle, \\
|Y^{(3)}\rangle &= [F](I-[A])|\psi_0\rangle \;=\; [F]\frac{1}{\sqrt{3}}(|B\rangle + |C\rangle) \;=\; 0, \\
|Y^{(4)}\rangle &= (I-[F])(I-[A])|\psi_0\rangle \;=\; (I-[F])\frac{1}{\sqrt{3}}(|B\rangle + |C\rangle) \;=\; \frac{1}{\sqrt{3}}(|B\rangle + |C\rangle).
\end{aligned}$$

Since $\Pr(Y) = \langle Y|Y\rangle$, we find $\Pr(Y^{(1)}) = 1/9$, $\Pr(Y^{(2)}) = 2/9$, $\Pr(Y^{(3)}) = 0$, $\Pr(Y^{(4)}) = 2/3$. Note that the probabilities sum to 1.

(b) Demonstrate that $|Y^{(1)}\rangle$ is consistent with $|Y^{(2)}\rangle$, $|Y^{(3)}\rangle$, and $|Y^{(4)}\rangle$.

Answer: Consistency requires $\langle Y^{(i)}|Y^{(j)}\rangle = 0$ for $i \neq j$. For $i = 1$ we have $\langle Y^{(1)}|Y^{(2)}\rangle = 0$ because $\langle A|F\rangle = 1/\sqrt{3}$, $\langle Y^{(1)}|Y^{(3)}\rangle = 0$ because $|Y^{(3)}\rangle = 0$, and $\langle Y^{(1)}|Y^{(4)}\rangle = 0$ because $[F]$ is orthogonal to $I - [F]$.

(c) Calculate the probability that the particle was in location A at t_1 given that it started in $|\psi_0\rangle$ at t_0 and ended in $|F\rangle$ at t_2. *Hint:* in which history(ies) do the various events occur at their respective times? Be sure to derive your answer algebraically using joint and conditional probabilities.

> **Answer:** The conditional probability for one event conditioned on two others is the ratio of the joint probability for the three events divided by the joint probability for the two conditions,
>
> $$\Pr\left([A_1] \mid [\psi_0] \wedge [F_2]\right) = \Pr\left([\psi_0], [A_1], [F_2]\right) / \Pr\left([\psi_0], [F_2]\right).$$
>
> The two conditions, and also the three events, occur with nonzero probability only in history $Y^{(1)}$. Hence, $\Pr\left([A_1] \mid [\psi_0] \wedge [F_2]\right) = \Pr\left(Y^{(1)}\right) / \Pr\left(Y^{(1)}\right) = 1$. For sure the particle was at location A at time t_1.

(d) An alternate family $\mathcal{B}$ utilizes property $[B]$ in place of $[A]$ at time t_1. You would find the same probability for location B at t_1 using family $\mathcal{B}$ as you obtained in part (c) for the probability of location A at t_1 using family $\mathcal{A}$. Nonetheless your answers (assuming you solved (c) correctly) are in conflict. Briefly explain how you can reconcile these results.

> **Answer:** All calculations follow identically, so we conclude that for sure the particle was at location B at time t_1! This conclusion is *not* inconsistent with our previous result because histories containing $[B]$ at t_1 cannot be discussed simultaneously with histories containing $[A]$ at t_1. For example, the history $[\psi_0] \odot [A] \odot [F]$ is inconsistent with $[\psi_0] \odot [B] \odot [F]$.

2. Matrix elements of X and P (adapted from Cohen-Tannoudji, # 2.8)

Consider the Hamiltonian of a particle in one dimension,

$$H = \frac{P^2}{2m} + V(X),$$

with eigenvectors and eigenvalues $H|\varphi_n\rangle = E_n|\varphi_n\rangle$. Demonstrate that matrix elements of X and P are proportional, *i.e.*

$$\langle\varphi_n|P|\varphi_{n'}\rangle = \alpha_{n,n'}\langle\varphi_n|X|\varphi_{n'}\rangle,$$

and find the coefficient of proportionality $\alpha_{n,n'}$. *Hint:* the commutator $[X, P^2] = 2i\hbar P$ may be helpful.

Answer:

Note that $\langle\varphi_n|XH|\varphi_{n'}\rangle = E_{n'}\langle\varphi_n|X|\varphi_{n'}\rangle$, while $\langle\varphi_n|HX|\varphi_{n'}\rangle = E_n\langle\varphi_n|X|\varphi_{n'}\rangle$. Since $[X, H] = (i\hbar P/m)$, we have

$$\langle\varphi_n|P|\varphi_{n'}\rangle = \frac{m}{i\hbar}(E_{n'} - E_n)\langle\varphi_n|X|\varphi_{n'}\rangle.$$