1. Two spin-half particles, $a$ and $b$ in the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ are in a normalized entangled state $|\psi_0\rangle$ at time $t = 0$. Assume time evolution, $T(t,t')$.

(a) Given a projector $A$ corresponding to an event $A$ at time $t_2 > t_0$, give a formula for the probability $\Pr(A)$.

(b) Let $A^+$ be the event $S_{az} = +1/2$ and time $t_2$, let the time evolution be trivial, $T(t,t') = I$, and take as initial condition

$$|\psi_0\rangle = \left( |z_a^+, z_b^-\rangle + |z_a^-, z_b^+\rangle + |z_a^-, z_b^+\rangle \right) / \sqrt{3}.$$

Express the projector $A^+$ as a tensor product operator on $\mathcal{H}_a \otimes \mathcal{H}_b$ and evaluate $\Pr(A^+)$. 
(c) Define the three time \((t_0 < t_1 < t_2)\) history family

\[
[\psi_0] \odot \{B^j\} \odot \{A^k\}
\]

where \(j\) and \(k\) take values \(\pm\). Let \(B^\pm\) correspond to the events \(S_{bx} = \pm 1/2\) at time \(t_1\) and \(A^\pm\) correspond to the events \(S_{az} = \pm 1/2\) at time \(t_2\), and let \(T(t, t') = I\) as before. You may take as given that this history family is consistent. Use joint and conditional probabilities to evaluate \(\Pr(B^+|A^+)\). Please be careful to keep straight \(x\) and \(z\) spins, and particles \(a\) and \(b\) \(\text{Hint: the answer is easy if you look carefully at } \langle A^+|\psi_0\rangle\).
2. Heisenberg relations for angular momentum components.

(a) The variance of observable $A$ is $(\Delta A)^2 \equiv \langle A^2 \rangle - \langle A \rangle^2$. Derive a Heisenberg-like bound for the product $\Delta J_x \Delta J_y$. \textit{Hint:} in class we derived $\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|$ for operators such that $[A, B] = iC$.

(b) Does a state exist for which all components of $J$ have simultaneous definite values? If so, give an example and verify it. If not, explain why not.
(c) Evaluate the product $(\Delta J_x)^2(\Delta J_y)^2$ in the $\{J^2, J_z\}$ eigenstate $|jm\rangle$. Briefly comment on the dependence on $m$ and give a geometrical interpretation.
3. Particle in a cylindrically symmetric potential (adapted from Cohen-Tannoudji, #7.1)

(a) Express the Hamiltonian of a particle in a cylindrically symmetric potential $V(r)$ using cylindrical coordinates $(r, \varphi, z)$ in terms of $L_z$, $P_z$, and differential operators in the variable $r$. You may wish to recall

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right), \quad L_z = \frac{\hbar}{i} \frac{\partial}{\partial \varphi}, \quad P_z = \frac{\hbar}{i} \frac{\partial}{\partial z}.$$ 

(b) Evaluate the commutators $[L_z, H]$ and $[P_z, H]$. 

(c) Write down eigenfunctions of $L_z$ and $P_z$. State their eigenvalues and any conditions those eigenvalues must obey. Use these eigenfunctions to give a general expression for $\psi_{nmk}(r, \varphi, z)$, a simultaneous eigenfunction of $H$, $L_z$ and $P_z$. Please leave the dependence of $\psi$ on $r$ in the form of an unspecified function $R(r)$. 


(d) Let $\Sigma_y = \Sigma_y^{-1}$ be a reflection in the $xz$ plane (i.e. $\Sigma_y : (x, y, z) = (x, -y, z)$). Show that the commutator $[\Sigma_y, H] = 0$ and the anticommutator $\{\Sigma_y, L_z\}$ vanish. Deduce that $\Sigma_y \psi_{nmk}(r, \varphi, z)$ is an eigenfunction of $L_z$ and determine its eigenvalue. What does this imply about degeneracies of the Hamiltonian $H$?