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33-755 Quantum I Final Exam Friday, Dec. 13, 2019

Every part of each problem is worth 20 points. Some parts can be solved independently of others.

1. Two spin-half particles, a and b in the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ are in a normalized entangled state $|\psi_0\rangle$ at time $t = 0$. Assume time evolution, $\mathcal{T}(t, t')$.

(a) Given a projector $\mathcal{A}$ corresponding to an event A at time $t_2 > t_0$, give a formula for the probability $\Pr(A)$.

(b) Let A^+ be the event $S_{az} = +1/2$ and time t_2, let the time evolution be trivial, $T(t, t') = I$, and take as initial condition

$$|\psi_0\rangle = \big(|z_a^+, z_b^+\rangle + |z_a^+, z_b^-\rangle + |z_a^-, z_b^-\rangle\big)/\sqrt{3}.$$

Express the projector $\mathcal{A}^+$ as a tensor product operator on $\mathcal{H}_a \otimes \mathcal{H}_b$ and evaluate $\Pr(A^+)$.

(c) Define the three time ($t_0 < t_1 < t_2$) history family

$$[\psi_0] \odot \{B^j\} \odot \{A^k\}$$

where j and k take values $\pm$. Let $B^\pm$ correspond to the events $S_{bx} = \pm 1/2$ at time t_1 and $A^\pm$ correspond to the events $S_{az} = \pm 1/2$ at time t_2, and let $T(t, t') = I$ as before. You may take as given that this history family is consistent. Use joint and conditional probabilities to evaluate $\Pr(B^+|A^+)$. Please be careful to keep straight x and z spins, and particles a and b! *Hint:* the answer is easy if you look carefully at $\mathcal{A}^+|\psi_0\rangle$.

2. Heisenberg relations for angular momentum components.

(a) The variance of observable A is $(\Delta A)^2 \equiv \langle A^2 \rangle - \langle A \rangle^2$. Derive a Heisenberg-like bound for the product $\Delta J_x \Delta J_y$. *Hint:* in class we derived $\Delta A \Delta B \geq \frac{1}{2} |\langle C \rangle|$ for operators such that $[A, B] = iC$.

(b) Does a state exist for which all components of $\mathbf{J}$ have simultaneous definite values? If so, give an example and verify it. If not, explain why not.

(c) Evaluate the product $(\Delta J_x)^2(\Delta J_y)^2$ in the $\{J^2, J_z\}$ eigenstate $|jm\rangle$. Briefly comment on the dependence on m and give a geometrical interpretation.

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3. Particle in a cylindrically symmetric potential (adapted from Cohen-Tannoudji, #7.1)

(a) Express the Hamiltonian of a particle in a cylindrically symmetric potential $V(r)$ using cylindrical coordinates (r, φ, z) in terms of L_z, P_z, and differential operators in the variable r. You may wish to recall

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right), \qquad L_z = \frac{\hbar}{i}\frac{\partial}{\partial \varphi}, \qquad P_Z = \frac{\hbar}{i}\frac{\partial}{\partial z}.$$

(b) Evaluate the commutators $[L_z, H]$ and $[P_z, H]$.

(c) Write down eigenfunctions of L_z and P_z. State their eigenvalues and any conditions those eigenvalues must obey. Use these eigenfunctions to give a general expression for $\psi_{nmk}(r, \varphi, z)$, a simultaneous eigenfunction of H, L_z and P_z. Please leave the dependence of ψ on r in the form of an unspecified function $R(r)$.

(d) Let $\Sigma_y = \Sigma_y^{-1}$ be a reflection in the xz plane (i.e. $\Sigma_y : (x, y, z) = (x, -y, z)$). Show that the commutator $[\Sigma_y, H] = 0$ and the anticommutator $\{\Sigma_y, L_z\}$ vanish. Deduce that $\Sigma_y \psi_{nmk}(r, \varphi, z)$ is an eigenfunction of L_z and determine its eigenvalue. What does this imply about degeneracies of the Hamiltonian H?