NAME: ______SOLUTIONS______

33-755 Quantum I    Final Exam    Friday, Dec. 13, 2019

Every part of each problem is worth 20 points. Some parts can be solved independently of others.

**1.** Two spin-half particles, $a$ and $b$ in the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ are in a normalized entangled state $|\psi_0\rangle$ at time $t = 0$. Assume time evolution, $\mathcal{T}(t, t')$.

*(a)* Given a projector $\mathcal{A}$ corresponding to an event $A$ at time $t_2 > t_0$, give a formula for the probability $\Pr(A)$.

**Answer:**

$$\Pr(A) = \langle\psi_0|T(t_0, t_2)\mathcal{A}T(t_2, t_0)|\psi_0\rangle$$

*(b)* Let $A^+$ be the event $S_{az} = +1/2$ and time $t_2$, let the time evolution be trivial, $T(t, t') = I$, and take as initial condition

$$|\psi_0\rangle = \left(|z_a^+, z_b^+\rangle + |z_a^+, z_b^-\rangle + |z_a^-, z_b^-\rangle\right)/\sqrt{3}.$$

Express the projector $\mathcal{A}^+$ as a tensor product operator on $\mathcal{H}_a \otimes \mathcal{H}_b$ and evaluate $\Pr(A^+)$.

**Answer:**

$$\mathcal{A}^+ = [z_a^+] \otimes I_b$$

$$\mathcal{A}^+ I|\psi_0\rangle = \left(|z_a^+, z_b^+\rangle + |z_a^+, z_b^-\rangle\right)$$

$$= \sqrt{\frac{2}{3}}|z_a^+, x_a^+\rangle$$

Squaring yields $\Pr(A) = 2/3$.

*(c)* Define the three time ($t_0 < t_1 < t_2$) history family

$$[\psi_0] \odot \{B^j\} \odot \{A^k\}$$

where $j$ and $k$ take values $\pm$. Let $B^{\pm}$ correspond to the events $S_{bx} = \pm 1/2$ at time $t_1$ and $A^{\pm}$ correspond to the events $S_{az} = \pm 1/2$ at time $t_2$, and let $T(t, t') = I$ as before. You may take as given that this history family is consistent. Use joint and conditional probabilities to evaluate $\Pr(B^+|A^+)$. Please be careful to keep straight $x$ and $z$ spins, and particles $a$ and $b$! *Hint:* the answer is easy if you look carefully at $\mathcal{A}^+|\psi_0\rangle$.

**Answer:** By laws of probability, joint and conditional probabilities are related by $\Pr(B^+|A^+) = \Pr(B^+, A^+)/\Pr(A^+)$. We already evaluated the denominator in part $b$. To evaluate the numerator, form the chain ket

$$\begin{aligned}
|(B^+ \odot A^+)\rangle &= \big([z_a^+] \otimes I_b\big)\big(I_a \otimes [x_b^+]\big)|\psi_0\rangle \\
&= \big([z_a^+] \otimes [x_b^+]\big)|\psi_0\rangle \\
&= \sqrt{\frac{2}{3}}|z_a^+, x_b^+\rangle
\end{aligned}$$

where the last step was observed in part $b$. Because the joint probability

$$\Pr(B^+, A^+) = \langle (B^+ \odot A^+)|(B^+ \odot A^+)\rangle = \frac{2}{3}$$

equals $\Pr(A^+)$ as obtained in $b$, the conditional probability

$$\Pr(B^+|A^+) = 1$$

**2.** Heisenberg relations for angular momentum components.

*(a)* The variance of observable $A$ is $(\Delta A)^2 \equiv \langle A^2\rangle - \langle A\rangle^2$. Derive a Heisenberg-like bound for the product $\Delta J_x \Delta J_y$. *Hint:* in class we derived $\Delta A \Delta B \geq \frac{1}{2}|\langle C\rangle|$ for operators such that $[A, B] = iC$.

**Answer:** Since $[J_x, J_y] = i\hbar J_z$, we immediately have $\Delta J_x \Delta J_y \geq \frac{\hbar}{2}\langle J_z\rangle$.

*(b)* Does a state exist for which all components of $\mathbf{J}$ have simultaneous definite values? If so, give an example and verify it. If not, explain why not.

**Answer:** Given (a) and its cyclic permutations we see we must have $\langle J_x\rangle = \langle J_y\rangle = \langle J_z\rangle = 0$. Let's guess that $|jm\rangle = |00\rangle$ has all vanishing variances. Check: $J_z|00\rangle = 0$ and $J_z^2|00\rangle = 0$, so $\Delta J_z = 0$. Since $J_x = (J_+ + J_-)/2$, and $J_\pm|00\rangle = 0$, we find $\Delta J_x = 0$ and similarly for $J_y = (J_+ - J_-)/2$.

*(c)* Evaluate the product $(\Delta J_x)^2(\Delta J_y)^2$ in the $\{J^2, J_z\}$ eigenstate $|jm\rangle$. Briefly comment on the dependence on $m$ and give a geometrical interpretation.

**Answer:** Since $\langle jm|J_x|jm\rangle = 0$, we have $(\Delta J_x)^2 = \langle jm|J_x^2|jm\rangle$, and similarly for $J_y$. By symmetry,

$$\langle jm|J_x^2|jm\rangle = \langle jm|J_y^2|jm\rangle = \frac{1}{2}\langle jm|(J^2 - J_z^2)|jm\rangle$$

where $J^2 = J_x^2 + J_y^2 + J_z^2$. Evaluating $J^2|jm\rangle$ and $J_z|jm\rangle$ we find

$$(\Delta J_x)^2(\Delta J_y)^2 = \frac{\hbar^2}{2}\left(j(j+1) - m^2\right)$$

The product is maximized when $|m|$ is minimized (*i.e.* $\mathbf{J}$ lies close to the $xy$-plane) and the product is minized when $m = \pm j$ (*i.e.* $\mathbf{J}$ points as far as possible from the $xy$-plane).

**3.** Particle in a cylindrically symmetric potential (adapted from Cohen-Tannoudji, #7.1)

*(a)* Express the Hamiltonian of a particle in a cylindrically symmetric potential $V(r)$ using cylindrical coordinates $(r, \varphi, z)$ in terms of $L_z$, $P_z$, and differential operators in the variable $r$. You may wish to recall

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right), \qquad L_z = \frac{\hbar}{i}\frac{\partial}{\partial \varphi}, \qquad P_Z = \frac{\hbar}{i}\frac{\partial}{\partial z}.$$

**Answer:**

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\nabla^2 + V(r) \\ &= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + V(r) + \frac{1}{2mr^2}L_z^2 + \frac{1}{2m}P_z^2 \end{aligned}$$

*(b)* Evaluate the commutators $[L_z, H]$ and $[P_z, H]$.

**Answer:**

$[L_z, H] = 0$ because $\varphi$ enters $H$ only via $L_z$.

$[P_z, H] = 0$ because $z$ enters $H$ only via $P_z$.

*(c)* Write down eigenfunctions of $L_z$ and $P_z$. State their eigenvalues and any conditions those eigenvalues must obey. Use these eigenfunctions to give a general expression for $\psi_{nmk}(r, \varphi, z)$, a simultaneous eigenfunction of $H$, $L_z$ and $P_z$. Please leave the dependence of $\psi$ on $r$ in the form of an unspecified function $R(r)$.

**Answer:** Eigenfunctions of $L_z$ are $\exp(im\varphi)$ with eigenvalue $m \in \mathbb{Z}$.

Eigenfunctions of $P_z$ are $\exp(ikz)$ with eigenvalue $k \in \mathbb{R}$.

Simultaneous eigenfunctions of $H$, $L_z$ and $P_z$ are given by products

$$\psi_{nmk}(r, \varphi, z) = R_{nk}(r) e^{im\varphi} e^{ikz}.$$

*(d)* Let $\Sigma_y = \Sigma_y^{-1}$ be a reflection in the $xz$ plane (i.e. $\Sigma_y : (x, y, z) = (x, -y, z)$). Show that the commutator $[\Sigma_y, H] = 0$ and the anticommutator $\{\Sigma_y, L_z\}$ vanish. Deduce that $\Sigma_y \psi_{nmk}(r, \varphi, z)$ is an eigenfunction of $L_z$ and determine its eigenvalue. What does this imply about degeneracies of the Hamiltonian $H$?

**Answer:** $\Sigma_y$ leaves $r$ and $z$ invariant while reversing the sign of $\varphi$. Hence $\Sigma_y H \Sigma_y^{-1} = H$, while $\Sigma_y L_z \Sigma_y^{-1} = -L_z$. Consider

$$\begin{aligned} H\left(\Sigma_y \psi_{n,m,k}\right) &= \Sigma_y H \psi_{n,m,k} \\ &= E_{n,m,k}\left(\Sigma_y \psi_{n,m,k}\right) \end{aligned}$$

But $\Sigma_y \psi_{nmk}(r, \varphi, z) = \psi_{n,-m,k}(r, \varphi, z)$ differs from $\psi_{nmk}$ whenever $m \neq 0$. Hence the eigenstates of $H$ are twofold degenerate for $m \neq 0$. Of course there may be additional degeneracies, e.g. $k \to -k$.