Mean occupation number for a harmonic oscillator of frequency ω in thermal equilibrium at temperature T

The density operator for a harmonic oscillator in thermal equilibrium is

$$\rho = \frac{1}{Z} e^{-H/k_B T},$$

where

$$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 X^2 = (N + \frac{1}{2})\hbar\omega$$

with $N = a^\dagger a$.

(a) Show that $e^{H/k_B T} a e^{-H/k_B T} = a e^{-\hbar\omega/k_B T}$, and hence $\langle a^\dagger a \rangle \equiv \mathrm{tr} \rho N = \langle a a^\dagger \rangle e^{-\hbar\omega/k_B T}$.

(b) Use the commutation relation $[a, a^\dagger] = 1$ to show that $\langle N \rangle = 1/(e^{\hbar\omega/k_B T} - 1)$