33-755 Sample History Solutions, 2020

1. Consider histories

$$X = [z+] \odot [x+] \odot [z-], \quad Y = [z+] \odot [x-] \odot [z-]$$

defined at times t_0, t_1 and t_2.

(a) Show that they are compatible.

Answer: X, and Y are orthogonal and hence compatible (commuting).

(b) Assume trivial time evolution $T = I$. Is the history family consistent?

Answer: No. Consider the chain kets

$$|X\rangle = |z-\rangle\langle z - |x+\rangle\langle x + |z+\rangle$$

$$|Y\rangle = |z-\rangle\langle z - |x-\rangle\langle x - |z+\rangle.$$

The inner product $\langle X|Y\rangle \neq 0$.

(c) Now let the spin evolve in the presence of a magnetic field along the y-axis. Assume the time evolution (expressed in the z-basis) is

$$T(t) = \begin{pmatrix} \cos\omega t/2 & -\sin\omega t/2 \\ \sin\omega t/2 & \cos\omega t/2 \end{pmatrix}$$

and define the three times as $t_0 = 0$, $t_1 = \pi/2\omega$ and $t_2 = \pi/\omega$. Are the histories X and Y consistent? Why or why not?

Answer: Let the time evolution operator

$$T_{10} = T_{21} = T \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Now the chain kets are

$$|X\rangle = |z-\rangle\langle z - |T|x+\rangle\langle x + |T|z+\rangle = |z-\rangle$$

$$|Y\rangle = |z-\rangle\langle z - |T|x-\rangle\langle x - |T|z+\rangle = |0\rangle.$$

Since $|Y\rangle = |0\rangle$, the inner product $\langle X|Y\rangle = 0$ so the pair is consistent.

(d) Assuming initial condition $|z+\rangle$, determine the probabilities $\Pr(X)$ and $\Pr(Y)$.

Answer: $\Pr(X) = 1$ and $\Pr(Y) = 0$.

2. Consider a random walk $\{s_j\}$ on the integers obeying $|s_{j+1} - s_j| \leq 1$ in which every history receives equal weight (i.e. $p = q = r = 1/3$). Suppose the initial probability distribution for s_0 is given by $\Pr(s_0 = 0) = 1/4$ and $\Pr(s_0 = 1) = 3/4$. Find the conditional probability $\Pr(s_2 = 1|s_3 = 2)$.

Answer: There are $3^3 = 27$ walks with nonzero probability for the time interval from $t = 0$ to $t = 3$ that start at a $s_0 = 0$. The probability of each walk, conditioned on $s_0 = 0$ is $1/27$. However, since we assume that $\Pr(s_0) = 1/4$, the probability of each is $(1/27)(1/4)$. Similarly, for the 27 walks that start at $s_0 = 1$ the probability is $(1/27)(3/4)$.

Now we include the second condition, that the walk ends at $s_3 = 2$. There are three such walks starting at $s_0 = 0$ and six such walks starting at $s_0 = 1$ for a total probability of $\Pr(s_3 = 2) = 3 \times (1/27)(1/4) + 6 \times (1/27)(3/4) = 7/36$. Of these (see table below), two walks pass from $s_0 = 0$ through $s_2 = 1$ contributing a probability of $2 \times (1/27)(1/4)$. Three walks pass from $s_0 = 1$ through $s_1 = 1$ contributing a probability $3 \times (1/27)(3/4)$. The total probability of $s_0 = 0$ or1, and $s_2 = 1$, and $s_3 = 2$, is the sum of contributions from $s_0 = 0$ and $s_0 = 1$, which is $(11/4)(1/27) = 11/108$. Divide by the probability of $s_3 = 2$ to obtain the conditional probability $\Pr(s_2 = 1|s_3 = 2) = (11/108)/(7/36) = 11/21$.

$s_0 = 0$:

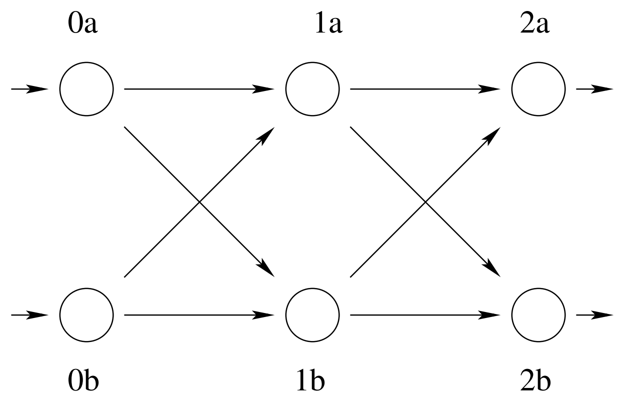
s_1	s_2	s_3
0	1	2
1	1	2
1	2	2

;

$s_0 = 1$:

s_1	s_2	s_3
0	1	2
1	1	2
1	2	2
2	1	2
3	2	2
2	3	2

,



3. The figure (following page) shows a toy model with shift operator

$$S|0a\rangle = \frac{1}{\sqrt{2}}(|1a\rangle + |1b\rangle), \quad S|1a\rangle = \frac{1}{\sqrt{2}}(|2a\rangle + |2b\rangle),$$

$$S|0b\rangle = \frac{1}{\sqrt{2}}(-|1a\rangle + |1b\rangle), \quad S|1b\rangle = \frac{1}{\sqrt{2}}(-|2a\rangle + |2b\rangle).$$

The histories of interest are from a family of the form

$$[\psi_0] \odot \{[1a], [1b], P\} \odot \{[2a], [2b], Q\}$$

at times $t = 0, 1, 2$, where $P = I - [1a] - [1b]$, $Q = I - [2a] - [2b]$.

(a) With time development operator T = S, show that if $[\psi_0] = [0a]$ this family is inconsistent.

Answer: Consider two chain kets,

$$|(0a, 1a, 2a)\rangle = [2a]S[1a]S|0a\rangle = \frac{1}{2}|2a\rangle$$

$$|(0a, 1b, 2a)\rangle = [2a]S[1b]S|0a\rangle = -\frac{1}{2}|2a\rangle.$$

They are not orthogonal, hence the family is not consistent.

(b) Find a $|\psi_0\rangle$ that is a superposition of $|0a\rangle$ and $|0b\rangle$, for which the family given above is consistent.

Answer: Try

$$|\psi_0\rangle = S^{-1}|1a\rangle = (|0a\rangle - |0b\rangle)/\sqrt{2},$$

since as a consequence $S|\psi_0\rangle = |1a\rangle$ which is orthogonal to $|1b\rangle$, hence the chain ket $|(\psi_0, 1b, 2a)\rangle$ vanishes, removing the inconsistency.

c) Now include a two-state measuring device $|n\rangle, n = 0$ or 1, and let $T = SR$, where $R = I \otimes I$ except for $R|1a, n\rangle = |1a, 1-n\rangle$. Indicate on the toy model figure where this detector is located. Look at histories of the form given above except that $[\psi_0]$ is replaced by $[\Psi_0]$, where $|\Psi_0\rangle = |\psi_0\rangle \otimes |n = 0\rangle$. What can you say about the consistency of families for different choices of the initial particle state $|\psi_0\rangle$? Again consider only cases in which $|\psi_0\rangle$ is a superposition of $|0a\rangle$ and $|0b\rangle$.

Answer: With a measuring device in place *any* choice of $|\psi_0\rangle$ in the subspace spanned by $|0a\rangle$ and $|0b\rangle$ can be used, and the family

$$[\Psi_0] \odot \{[1a], [1b], \tilde{P}\} \odot \{[2a], [2b], \tilde{Q}\}$$

will be consistent. For example, consider the chain kets

$$|(\Psi_0, 1a, 2a)\rangle = [2a]SR[1a]SR|\Psi_0\rangle = (\langle 2a|S|1a\rangle\langle 1a|S|\psi_0\rangle)[|2a\rangle \otimes |n = 1\rangle]$$

$$|(\Psi_0, 1b, 2a)\rangle = [2a]SR[1b]SR|\Psi_0\rangle = (\langle 2a|S|1b\rangle\langle 1b|S|\psi_0\rangle)[|2a\rangle \otimes |n = 0\rangle]$$

Consistency follows because of the orthogonality of the detector states.