Exam 1 further examples

Spin 1 particle

Consider a spin-1 particle with basis states \{\text{\text{|+\rangle}}, \text{\text{|0\rangle}}, \text{\text{|-\rangle}}\} and Hamiltonian.

\[
H = \frac{\hbar \omega}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

a) A solution to Schrödinger’s equation \(i\hbar \frac{d}{dt} |\psi_t\rangle = H |\psi_t\rangle\) takes the form

\[|\psi_t\rangle = c_+ (t) \text{\text{|-\rangle}} + c_0 (t) \text{\text{|0\rangle}} + c_- (t) \text{\text{|+\rangle}}.\]

Write down a set of coupled differential equations satisfied by the complex coefficients \(c_+ (t), c_0 (t), \text{\text{and}} c_- (t)\).

b) Assuming \(\langle \psi_t | \psi_t \rangle = 1\), find the probability in terms of the \(c_j (t)\) that if an appropriate measurement is carried out at time \(t\), the system will be found in the state \(|\phi\rangle\), for

(i): \(|\phi\rangle = |0\rangle\),
(ii): \(|\phi\rangle = (|+\rangle - i |-\rangle) / \sqrt{2}\).

Do NOT solve for \(c_j (t)!\)

c) A formal solution to Schrödinger’s equation can be written in the form

\[|\psi_t\rangle = T(t) |\psi_0\rangle, \text{\text{with}} \ T(t) = e^{-iHt/\hbar}.\]

Express \(T(t)\) as a 3 × 3 matrix in the usual \{\text{\text{|+\rangle}}, \text{\text{|0\rangle}}, \text{\text{|-\rangle}}\} basis. Hint: the three eigenvectors of \(H\) are \(|a, b\rangle = (1/2, \pm 1/\sqrt{2}, 1/2)\) and \(|c\rangle = (1, 0, -1)/\sqrt{2}.\)
Projectors on tensor product space of deuteron and electron spins in deuterium

The Hilbert space for the spin degrees of freedom of a deuterium atom is of the form $\mathcal{H}_e \otimes \mathcal{H}_d$, where $\mathcal{H}_e$ is the two-dimensional spin space of the electron, and $\mathcal{H}_d$ the three-dimensional space of the deuteron. Let $|+\rangle$ and $|−\rangle$ be an orthonormal basis of $\mathcal{H}_e$ with

$$S_z |+\rangle = +\frac{1}{2} |+\rangle, \quad S_z |−\rangle = −\frac{1}{2} |−\rangle,$$

and $|+\rangle$, $|0\rangle$, and $|−\rangle$ an orthonormal basis of $\mathcal{H}_d$ with

$$I_z |+\rangle = |+\rangle, \quad I_z |0\rangle = 0, \quad I_z |−\rangle = −|−\rangle.$$

Define the three projectors

$$P = |−0\rangle\langle −0|,$$
$$Q = (|+−\rangle\langle +−|) + (|+0\rangle\langle +0|) + (|−−\rangle\langle −−|),$$
$$R = (|+−\rangle\langle +−|) + (|−−\rangle\langle −−|),$$

where $|−0\rangle$ stands for $|−\rangle \otimes |0\rangle$, etc.

a) What is the dimension of $\mathcal{H} = \mathcal{H}_e \otimes \mathcal{H}_d$?

b) Can any of these projectors be written in the form $A \otimes I$ or $I \otimes B$, where $I$ denotes the identity operator? If so, find the corresponding $A$ or $B$.

c) Provide a physical interpretation of each of the projectors $P$, $Q$, and $R$ in terms of $S_z$ or $I_z$ or both.

d) Find the projectors $\tilde{Q}$ and $\tilde{R}$ corresponding to the negations of the projectors $Q$ and $R$, and again give a physical interpretation.

e) Find the projector $K$ on $\mathcal{H}_e \otimes \mathcal{H}_d$ that corresponds to the property $S_y = +1/2$. You may use the fact that for a spin-half particle the state $|+\rangle + i|−\rangle$ is an eigenstate of $S_y$ with eigenvalue $+1/2$. Expressing $K$ as a sum of dyads in a similar form to $P$, $Q$ and $R$ above leads to a rather lengthy expression, and you do not need to write it out as long as you make it plain that you know what you are doing.

f) With which of the projectors $P$, $Q$, and $R$ does $K$ commute? Discuss this in terms of compatible and incompatible properties.
g) The smallest Boolean algebra containing both $Q$ and $R$ is generated by a particular decomposition of the identity on $\mathcal{H}_e \otimes \mathcal{H}_d$. What is this decomposition of the identity?

**Commuting observables, Sample spaces and probabilities**

In a particular orthonormal basis the matrix for the observable $V$ is

$$
\begin{pmatrix}
0 & 1 & i \\
1 & -1/2 & -i/2 \\
-i & i/2 & -1/2
\end{pmatrix}
$$

a) Find the corresponding quantum sample space, i.e., decomposition of the identity as a sum of three projectors $P_1$, $P_2$, and $P_3$; express each projector as a $3 \times 3$ matrix in this basis. [Hint. Feel free to use a computer program to diagonalize the matrix.]

b) Suppose that $\langle V \rangle = 0$, $\langle V^2 \rangle = 6/5$. What probabilities should be assigned to each element of the sample space? Be sure and explain what you are doing. Then compute $\langle V^3 \rangle$.

c) Here is the matrix of another observable $W$ that commutes with $V$.

$$
\begin{pmatrix}
4/3 & 5/3 & 5i/3 \\
5/3 & 4/3 & -5i/3 \\
-5i/3 & 5i/3 & 4/3
\end{pmatrix}
$$

Find the sample space, which in this case consists of two projectors $Q_1$ and $Q_2$, where $Q_1$ projects on a one-dimensional space and $Q_2$ on a two-dimensional space. Assign probabilities of $2/5$ and $3/5$ to the $Q_1$ and $Q_2$ subspaces, respectively, and compute $\langle W \rangle$ and $\langle W^2 \rangle$. (Note: These probabilities have nothing to do with those in (b).)

d) Even if the probability distribution is not given in advance, it is possible to sensibly discuss certain assertions of the form “If $W = b$, then the probability that $V = a$ is . . .,” where . . . might be “one” or “zero” or “unknown.” Which framework (sample space) should you use in order to discuss assertions of this form in a meaningful way? Work out examples of sensible statements of this type for various choices of $b$ and $a$. 