33-755 Homework 1

1. Find the eigenvalues $\lambda_{1,2}$ of the Hermitian matrix

$$F = \begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}$$

along with the normalized column vectors $|\psi_{1,2}\rangle$. Use the latter to construct the projectors $P_1 = |\psi_1\rangle\langle\psi_1|$ and $P_2 = |\psi_2\rangle\langle\psi_2|$ as 2×2 matrices. Use matrix addition to check that

$$I = P_1 + P_2 \quad F = \lambda_1 P_1 + \lambda_2 P_2.$$

2. Show that for a spin half particle $[z^+]$ and $[x^+]$ do not commute, and then give an argument why the same will be true for the projectors $[v^+]$ and $[w^+]$ for the spin to be along any two directions v and w (unit vectors on the sphere), apart from certain exceptional cases, which you should specify.

3. The matrices

$$R = \frac{1}{3}\begin{pmatrix} 2 & \text{-i} & \text{i} \\ \text{i} & 2 & 1 \\ \text{-i} & 1 & 2 \end{pmatrix} \quad \text{and} \quad S = \frac{1}{6}\begin{pmatrix} 5 & \text{2i} & \text{i} \\ \text{-2i} & 2 & -2 \\ \text{-i} & -2 & 5 \end{pmatrix}$$

represent two commuting projectors on a three-dimensional Hilbert space H.

a) What are the dimensions of the subspaces that R and S project into?

b) Find the decomposition $\{P_j\}$ of the identity I such that R and S belong to the corresponding Boolean event algebra. That is to say, they can be written as sums of one or more of the one-dimensional P_j. Express your answer in terms of 3x3 matrices using the same basis as for R and S above. Note: If B and C are members of the Boolean event algebra of projectors, then so are BC, $I - B$, and $I - C$.

4. a) Let S and T be two indicator functions on the classical phase space, and suppose that $ST = T$. One of the following assertions can be proved.

(i) If the system possesses property S it necessarily also possesses property T.

(ii) If the system possesses property T it necessarily also possesses property S.

Decide which is correct and explain why (a formal proof is not needed).

b) For a 3-dimensional Hilbert space construct two projectors P and Q such that $PQ \neq QP$, but the subspace $\mathcal{R} = \mathcal{P} \cap \mathcal{Q}$ is of dimension 1 or more. Try and make it a very simple example. Let R be the projector onto $\mathcal{R}$. What are PR and QR? [Hint 1: Both are projectors. Hint 2: You may prefer to work backwards, starting from R or $\mathcal{R}$.]

c) Assuming that classical mechanics, i.e. (a), is a suitable guide, what might you conclude about relationships of the properties of a quantum system on the basis of what you found in (b)? Why might this conclusion be somewhat troubling in light of the fact that $PQ \neq QP$?

5. Consider a composite system of two spin-1/2 particles, taking the $S_z^{(i)}$ basis set $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$, in that order. The total spin operator $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ has matrix representation

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{2}\begin{pmatrix} 0 & -i & -i & 0 \\ i & 0 & 0 & -i \\ i & 0 & 0 & -i \\ 0 & i & i & 0 \end{pmatrix},$$

$$S_z = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad S^2 = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(a) From the set of operators $\{S_x, S_y, S_z, S^2\}$ select a complete set of commuting observables (CSCO, see Cohen-Tannoudji, chapter II). How many different choices can you find?

(b) Find a set of four matrices constituting a decomposition of the identity that includes projectors onto eigenstates of S_x.

(c) Express S^2 in terms of the matrices found in part (b).

(d) Define the states

$$|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad |\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad |\chi\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and apply the Born rule to:

(*i*) Find the average values of S_x, S_y, S_z and S^2 in the state $|\psi\rangle$.

(*ii*) Determine the probability that $S^2 = 0$ in the state $|\phi\rangle$.

(*iii*) Determine the probability that $S_x = 1$ in the state $|\chi\rangle$.