33-755 Homework 2

1. Mixed state density operators for a spin 1/2 particle

a) Represent the properties $P_{[z+]}$ and $P_{[x+]}$ as projector matrices using the basis set $\{|z+\rangle, |z-\rangle\}$. Verify that these projectors are pure state density matrices.

b) Imagine filling a jar with a mixture of particles individually known to be in state $[z+]$ with probability $p_{z+} = 3/4$, or in state $[x+]$ with probability $p_{x+} = 1/4$. A mixed state is created by randomly drawing particles from the jar. Express this mixed state as a density matrix in the basis set $\{|z+\rangle, |z-\rangle\}$. Verify that your result *is* a mixed state density matrix.

c) Use the mixed state density matrix to calculate, if you can, the following probabilities (if you can't, explain why not):

i) Pr $(S_z = +\hbar/2)$

ii) Pr $(S_x = +\hbar/2)$

iii) Pr $((S_z = +\hbar/2)$ **or** $(S_x = +\hbar/2))$

2. Partial trace of two-spin density operator

Consider a system of two spin-half particles a and b, in Hilbert spaces $\mathcal{H}_a$ and $\mathcal{H}_b$, and use the notation $|+-\rangle$ for $|z_a^+\rangle \otimes |z_b^-\rangle$, etc. The entangled "EPR" state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle \;-\; |-+\rangle\right).$$

(a) Express this state as a density operator $\rho_{ab} = |\psi\rangle\langle\psi|$.

(b) Calculate Tr ρ_{ab} and Tr ρ_{ab}^2 to verify that ρ_{ab} is indeed a density operator and that it represents a pure state.

(c) Calculate the *reduced* density operator $\rho_a \equiv \mathrm{Tr}_b\ \rho_{ab}$ (see Griffiths, ch. 6.5).

(d) Calculate Tr ρ_a and Tr ρ_a^2. Is ρ_a a density operator? Is it a pure state?

(e) Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|++\rangle \;+\; |-+\rangle\right).$$

Is $|\psi\rangle$ entangled? If not represent it as a product. Repeat (a-d) for the state $|\psi\rangle$.

3. Define the state (in the z-basis)

$$|\psi_0\rangle = |++\rangle \ + \ 3|+-\rangle \ - \ 3|-+\rangle \ - \ |--\rangle.$$

a) Find the norm $\|\psi_0\|$. Also write down a nonzero $|\phi\rangle$ which is orthogonal to $|\psi_0\rangle$.

b) Find a projector P of the form

$$P = A_1 \otimes B_1 + A_2 \otimes B_2,$$

where A_1 and A_2 are projectors on one-dimensional subspaces of $\mathcal{H}_a$, B_1 and B_2 projectors on one-dimensional subspaces of $\mathcal{H}_b$, chosen in such a way that $|\psi_0\rangle$ has property P. You may express these projectors in any way that is convenient, and your task may be simpler if we tell you that

$$|\psi_0\rangle = c_0|x_a^+\rangle \otimes |x_b^-\rangle \ + \ c_1|x_a^-\rangle \otimes |x_b^+\rangle$$

for an appropriate choice of nonzero constants c_0 and c_1. (You do *not* have to derive this expression!) Give a brief explanation of how you know that a system in the state $|\psi_0\rangle$ has the property corresponding to this projector P.

c) Suppose that $|\psi_0\rangle$ evolves unitarily from a time t_0 to a time t_1, where

$$|\psi_1\rangle = T(t_1, t_0)|\psi_0\rangle = |++\rangle \ - \ 3|+-\rangle \ + \ 3|-+\rangle \ - \ |--\rangle.$$

Assume $T(t_1, t_0)$ is given by a diagonal matrix in the $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ basis (rows and columns in this order). Find the matrix $T(t_1, t_0)$. Next suppose that this $T(t_1, t_0)$ results from a time-independent Hamiltonian H. Find the matrix for a Hamiltonian (in this same basis) that gives rise to $T(t_1, t_0)$. Express your answer in terms of $\tau = t_1 - t_0$ and $\hbar$ (which you can set equal to 1 if you prefer). Comment on whether your result for H is unique.

4. Schrödinger coupled differential equations and their relation to $T(t,t')$

a) Write the Schrödinger equation for $\psi(t) = \begin{pmatrix} c_0(t) \\ c_1(t) \end{pmatrix}$ given a Hamiltonian $H = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ as a pair of coupled differential equations for $c_0(t)$ and $c_1(t)$, assuming $\hbar = 1$ (A and B are real numbers and C is complex). Insert $c_0(t) = ae^{i\mu t}$ and $c_1(t) = be^{i\mu t}$ into these equations to find the allowed values of μ. How are the allowed values related to properties of the Hamiltonian? (You are not asked to find an explicit solution to the equations.)

b) Suppose you were given two sets of solutions to the differential equations: the pair $\{\hat{c}_0(t),\ \hat{c}_1(t)\}$ such that $\hat{c}_0(0) = 1,\ \hat{c}_1(0) = 0$; and the pair $\{\check{c}_0(t),\ \check{c}_1(t)\}$ with the property that $\check{c}_0(0) = 0,\ \check{c}_1(0) = 1$. These are linear combinations of your previous solutions, but you needn't determine them explicitly. Show how you can use these to construct the matrix for the time development operator $T(t-t')$; note that $T(t,t')$ depends only on the difference $t-t'$, because H is independent of time.

c) Replace C and C^* in H with time-dependent coefficients $C = \gamma e^{i\omega_0 t}$, $C^* = \gamma^* e^{-i\omega_0 t}$, where γ is a constant complex number. Show that a simple modification of the ansatz used in (a) allows a similar analysis of the differential equations in terms of suitable complex exponential functions of t. You are asked to find the exponents (frequencies), not work out the explicit solutions.

d) Suppose that for the modified differential equations using the time-dependent Hamiltonian in (c), or some other time-dependent Hamiltonian, you are again given two sets of solutions: $\{\hat{c}_0(t),\ \hat{c}_1(t)\}$ with $\hat{c}_0(0) = 1,\ \hat{c}_1(0) = 0$; and $\{\check{c}_0(t),\ \check{c}_1(t)\}$ with $\check{c}_0(0) = 0,\ \check{c}_1(0) = 1$. Because the Hamiltonian is time dependent the time development operator $T(t,t')$ no longer depends solely on the difference $t-t'$, but show that you can still write it, for general t and t' in terms of these special solutions. Hint: Think about expressing $T(t,t')$ as a product of two operators.