1. Spin states on the Bloch sphere

The unit vector $\mathbf{n}$ with polar angle $\theta$ and azimuthal angle $\varphi$ has Cartesian coordinates $(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Let $S_x = (\hbar/2)\sigma_x$, $S_y = (\hbar/2)\sigma_y$, and $S_z = (\hbar/2)\sigma_z$ be the $x$, $y$, and $z$ components of the spin operator $\mathbf{S}$.

a) Express the operator $S_\mathbf{n} \equiv \mathbf{n} \cdot \mathbf{S}$ as a matrix in the $\{z^\pm\}$ basis.

b) $S_\mathbf{n}$ has two eigenvectors that we shall denote $|\mathbf{n}^+\rangle$ and $|\mathbf{n}^-\rangle$. Determine these eigenvectors and their corresponding eigenvalues.

2. Spin in constant magnetic field

The following problem is adapted from Cohen-Tannoudji, chapter 4. Consider a spin 1/2 particle in a constant magnetic field tilted slightly away from the $z$-axis in the $yz$-plane. The Hamiltonian

$$H = \frac{1}{2} (\omega_y \sigma_y + \omega_z \sigma_z).$$

a) Express $H$ as a matrix in the standard basis $\{|z^+\rangle, |z^-\rangle\}$.

b) Express the eigenvalues and eigenvectors of $H$ as explicit superpositions within the standard basis.

c) On a single set of axes, sketch the eigenvalues of $H$ as functions of $\omega_z$ (consider both positive and negative values). Include both the case $\omega_y = 0$ and the case $\omega_y > 0$.

d) Expand the eigenvalues as power series in $\omega_z$, keeping only the lowest nonzero power of $\omega_z$. Treat both the case $\omega_y = 0$ and the case $\omega_y > 0$.

e) Letting the initial state $|\psi_0\rangle = |z^+\rangle$, determine the time-evolved state $|\psi(t)\rangle$. Hint: it may be convenient to express $|\psi(t)\rangle$ in the basis of eigenstates of the Hamiltonian.

f) Calculate the probability $P_{z^+}(t)$ of spin down $|z^-\rangle$ at time $t$ given the initial spin up state $|z^+\rangle$ at time 0. Sketch this probability over one full period of Rabi oscillation. Be sure to label your axes with characteristic times and probabilities.
3. Quantum gates

a) Let $Q$ be a spin 1/2 particle with magnetic moment $\mu = \gamma S = (\hbar/2)\gamma \sigma$. A static magnetic field of strength $B$ acts in the direction $\hat{n}$ for a period of time $\tau$. Determine the direction $\hat{n}$ and the time interval $\tau_H$ such that the unitary time evolution operator (expressed in the $\{|z^\pm\}\$ basis) is

$$\mathcal{H} = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$  

How does this transformation, known (aside from an irrelevant factor of $i$) as the Hadamard transformation, act on the states $\{|x^\pm\}$, $\{|y^\pm\}$, and $\{|z^\pm\}$ of $Q$? What happens if you evolve for time $2\tau_H$?

b) Let $Q_1$ and $Q_2$ be identical spin 1/2 particles. They interact with a magnetic field in the $\hat{z}$ direction, and with each other, for a duration $\tau$ under the Hamiltonian

$$H = \frac{1}{2}(\hbar \omega_0) \left( -I + \sigma_2^{(1)} + \sigma_2^{(2)} - \sigma_2^{(1)} \otimes \sigma_2^{(2)} \right).$$

Here superscripts indicate the individual spins, and tensor products with identity operators are implicit where needed. Determine the time interval $\tau_Z$ for which the unitary time evolution operator (expressed in the basis $\{|z^\pm z^\pm\}\$) is

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$  

This transformation is known as a controlled $Z$ gate.

c) In the tensor product space of $Q_1$ and $Q_2$, apply the sequence of operations: $\mathcal{H}$ on $Q_2$, followed by $Z$ on $Q_1 \otimes Q_2$, followed by $\mathcal{H}^\dagger$ on $Q_2$. Express the combined action as a matrix taking the basis states in the following order $\{|z^+ z^+, z^+ z^-, z^- z^+, z^- z^-\}\$. Why do we name this combination “controlled not”?

This question was inspired by reading a recent *Physics Today* article “Quantum computing with semiconductor spins” (see https://doi.org/10.1063/PT.3.4270) and was designed with help from Vikesh Siddhu.