33-755 Homework 3

1. Spin states on the Bloch sphere

The unit vector $\mathbf{n}$ with polar angle θ and azimuthal angle φ has Cartesian coordinates $(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Let $S_x = (\hbar/2)\sigma_x$, $S_y = (\hbar/2)\sigma_y$, and $S_z = (\hbar/2)\sigma_z$ be the x, y, and z components of the spin operator $\mathbf{S}$.

a) Express the operator $S_{\mathbf{n}} \equiv \mathbf{n} \cdot \mathbf{S}$ as a matrix in the $\{z^\pm\}$ basis.

b) $S_{\mathbf{n}}$ has two eigenvectors that we shall denote $|\mathbf{n}^+\rangle$ and $|\mathbf{n}^-\rangle$. Determine these eigenvectors and their corresponding eigenvalues.

2. Spin in constant magnetic field

The following problem is adapted from Cohen-Tannoudji, chapter 4. Consider a spin 1/2 particle in a constant magnetic field tilted slightly away from the z-axis in the yz-plane. The Hamiltonian

$$H = \frac{1}{2}\left(\omega_y \sigma_y + \omega_z \sigma_z\right).$$

a) Express H as a matrix in the standard basis $\{|z^+\rangle, |z^-\rangle\}$.

b) Express the eigenvalues and eigenvectors of H as explicit superpositions within the standard basis.

c) On a single set of axes, sketch the eigenvalues of H as functions of ω_z (consider both positive and negative values). Include both the case $\omega_y = 0$ and the case $\omega_y > 0$.

d) Expand the eigenvalues as power series in ω_z, keeping only the lowest nonzero power of ω_z. Treat both the case $\omega_y = 0$ and the case $\omega_y > 0$.

e) Letting the initial state $|\psi_0\rangle = |z^+\rangle$, determine the time-evolved state $|\psi(t)\rangle$. Hint: it may be convenient to express $|\psi(t)\rangle$ in the basis of eigenstates of the Hamiltonian.

f) Calculate the probability $P_{-+}(t)$ of spin down $|z^-\rangle$ at time t given the initial spin up state $|z^+\rangle$ at time 0. Sketch this probability over one full period of Rabi oscillation. Be sure to label your axes with characteristic times and probabilities.

3. Quantum gates

a) Let Q be a spin 1/2 particle with magnetic moment $\boldsymbol{\mu} = \gamma \mathbf{S} = (\hbar/2)\gamma\boldsymbol{\sigma}$. A static magnetic field of strength B acts in the direction $\hat{\mathbf{n}}$ for a period of time τ. Determine the direction $\hat{\mathbf{n}}$ and the time interval $\tau_{\mathcal{H}}$ such that the unitary time evolution operator expressed in the $\{|z^{\pm}\rangle\}$ basis (up to an irrelevant complex phase factor) is

$$\mathcal{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

How does this transformation, known as the Hadamard transformation, act on the states $\{|x^{\pm}\rangle\}$, $\{|y^{\pm}\rangle\}$, and $\{|z^{\pm}\rangle\}$ of Q? What happens if you evolve for time $2\tau_{\mathcal{H}}$?

b) Let Q_1 and Q_2 be identical spin 1/2 particles. They interact with a magnetic field in the $\hat{z}$ direction, and with each other, for a duration τ under the Hamiltonian

$$H_{\mathcal{Z}} = \frac{1}{2}(\hbar\omega_0)\left(-I + \sigma_z^{(1)} + \sigma_z^{(2)} - \sigma_z^{(1)} \otimes \sigma_z^{(2)}\right) .$$

Here superscripts indicate the individual spins, and tensor products with identity operators are implicit where needed. Determine the time interval $\tau_{\mathcal{Z}}$ for which the unitary time evolution operator (expressed in the basis $\{|z^{\pm}z^{\pm}\rangle\}$) is

$$\mathcal{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

This transformation is known as a controlled Z gate.

c) In the tensor product space of Q_1 and Q_2, apply the sequence of operations: $\mathcal{H}$ on Q_2, followed by $\mathcal{Z}$ on $Q_1 \otimes Q_2$, followed by $\mathcal{H}^{\dagger}$ on Q_2. Express the combined action as a matrix taking the basis states in the following order $\{|z^+z^+\rangle, |z^+z^-\rangle, |z^-z^+\rangle, |z^-z^-\rangle\}$. Why do we name this combination "controlled not"?

This question was inspired by a *Physics Today* article "Quantum computing with semiconductor spins" (see https://doi.org/10.1063/PT.3.4270)

4. The "Monty Hall" problem

Suppose you are a guest on the game show "Let's Make a Deal", hosted by Monty Hall. The game consists of 4 steps.

0. A car is randomly and secretly placed behind one of three doors; behind the other two are goats.

1. You pick a door (say, No. 1 as an example).

2. The host, who knows where the car is placed opens another door (say, No. 3), behind which is a goat.

3. The host asks "Do you want to switch your choice?" Is it to your advantage to change (say, pick door No. 2)?

Analyze this question by defining a stochastic process on a time sequence of sample spaces. Calculate the probability that you win a car, conditioned on choosing to switch.