33-755 Homework 5

1. Destructive measurement

Let $\{|s^k\rangle, k = 1, ..., N\}$ be a complete orthogonal basis set for the Hilbert space of a particle, $\mathcal{H}_p$. We also have a measuring device M with its own very high dimensional Hilbert space $\mathcal{H}_m$, initially in the ready state M_0. Basis states evolve along with the measuring device according to

$$(|s^k\rangle \otimes M_0) \to (|s^k\rangle \otimes M_1) \to |\mathcal{N}_2^k\rangle$$

where $\{|\mathcal{N}^k\rangle\}$ is a set of "pointer states" consisting of N different entangled states of the particle and measuring device. Think of each pointer state as corresponding to the needle on a meter pointing in one of N possible directions $\hat{\mathbf{n}}^k$ that can be observed and reported. The measurement is *destructive* because the state of the particle is entangled with the detector.

(a) Let the initial state of the particle be $|\psi_0\rangle = \sum_k c_k|s^k\rangle$, with $\sum_k |c_k|^2 = 1$, and consider histories of the type $\{Y^{jk} = ([\psi_0] \otimes M_0) \odot ([s^j] \otimes M_1) \odot ([\mathcal{N}_2^k])\}$. Demonstrate that these histories are compatible and consistent with each other.

(b) Determine the probability to end in the k^{th} pointer state.

(c) Determine the conditional probability the particle was in state j at time 1 given the pointer state k at time 2.

2. *Non*destructive measurement

The setup is the same as in question 1 except the particle is not destroyed by the measurement. The basis states evolve according to

$$(|s^k\rangle \otimes M_0) \to (|s^k\rangle \otimes M_1) \to (|s^k\rangle \otimes |M_2^k\rangle)$$

where the $|M^k\rangle$ states are states of the measuring apparatus only. As before, the initial particle state is $|\psi_0\rangle = \sum_k c_k|s^k\rangle$, with $\sum_k |c_k|^2 = 1$.

(a) Let $|\Psi_0\rangle = |\psi_0\rangle \otimes M_0$ be the initial state at time 0. Determine the final state $|\Psi_2\rangle$.

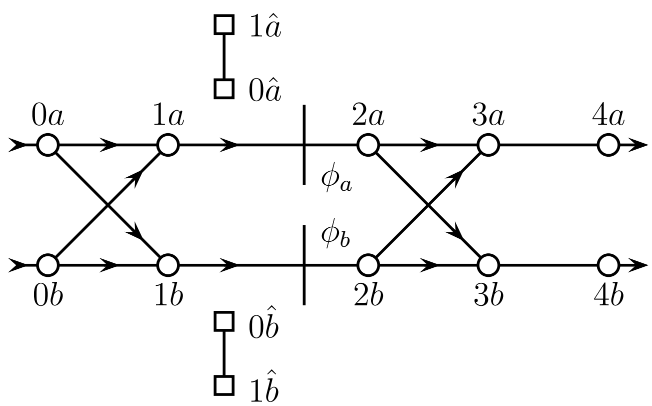
(b) $\rho_{\text{ini}} = |\psi_0\rangle\langle\psi_0|$ is the density operator for the initial pure state of the particle. What is the reduced density operator for the state of the particle subsequent to the measurement, irrespective of the measurement outcome? Under what conditions is the state pure or mixed?

(c) Assume that the measurement outcome is $|M^k\rangle$ for a specific k. What is the reduced density operator for the particle in this case? *Hint:* Remember that the probabilities that weight pure state components of a mixed state density operator reflect our knowledge of the state.

(d) Extend the unitary time evolution from time 0 up to time 3, subsequent to the nondestructive measurement. The time evolution operator from time 2 to 3 is the identity for both the particle and the measuring apparatus. Determine the conditional probability for state $|s^j\rangle$ at time 3 subject to the measurement M^k at time 2.

[**3.**] Consistent family for weak detection.

(a) Verify that the following figure is equivalent to the Mach-Zehnder interferometer, with the crossed arrows representing the beam splitters ($e.g.$ $S|0a\rangle \to (|1a\rangle + |1b\rangle)/\sqrt{2}$), *etc.*.



We will supplement the interferometer with weak detectors $\hat{a}$ and $\hat{b}$, and phase shifters ϕ_a and ϕ_b between positions 1 and 2. Thus the state $|1a, 0\hat{a}, 0\hat{b}\rangle \to e^{i\phi_a}(\alpha|2a, 0\hat{a}, 0\hat{b}\rangle + \beta|2a, 1\hat{a}, 0\hat{b}\rangle)$, while $|1b, 0\hat{a}, 0\hat{b}\rangle \to e^{i\phi_b}(\alpha|2b, 0\hat{a}, 0\hat{b}\rangle + \beta|2b, 0\hat{a}, 1\hat{b}\rangle)$.

(b) Verify, as claimed in class, that $\Pr\,([3a]_3) = |\alpha|^2 \sin^2(\Delta/2) + |\beta|^2/2$. Note that $|\Psi_0\rangle = |0a, 0\hat{a}, 0\hat{b}\rangle$, and $\Delta = \phi_a - \phi_b$.

(c) Demonstrate that the following 12 histories are mutually consistent. Give an interpretation to the meaning of the set $\{Y^{(a,b;\pm;\hat{a},\hat{b})}\}$ and of the set $\{Y^{(\pm;a,b)}\}$. Note, $|\chi\pm\rangle \equiv (|1a\rangle \pm |1b\rangle)/\sqrt{2}$ and $|3\pm\rangle \equiv (\pm|3a\rangle + |3b\rangle)/\sqrt{2}$. As usual, some identity operators are implicit.

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\begin{aligned}
Y^{(a,b;\pm;\hat{a},\hat{b})} &\equiv [\Psi_0] \;\odot\; \{[1a],[1b]\} \;\odot\; I \;\odot\; ([3\pm] \otimes \{[1\hat{a}],[1\hat{b}]\}) \\
Y^{(\pm;a,b)} &\equiv [\Psi_0] \;\odot\; \{[\chi\pm]\} \;\odot\; I \;\odot\; (\{[3a],[3b]\} \otimes [0\hat{a}] \otimes [0\hat{b}])
\end{aligned}
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