33-755 Homework 6

1. Bell states and dense coding

The Bell states defined by

$$|B^0\rangle = \big(|00\rangle + |11\rangle\big)/\sqrt{2}, \quad |B^1\rangle = \big(|01\rangle + |10\rangle\big)/\sqrt{2},$$
$$|B^2\rangle = \big(|00\rangle - |11\rangle\big)/\sqrt{2}, \quad |B^3\rangle = \big(|01\rangle - |10\rangle\big)/\sqrt{2}$$

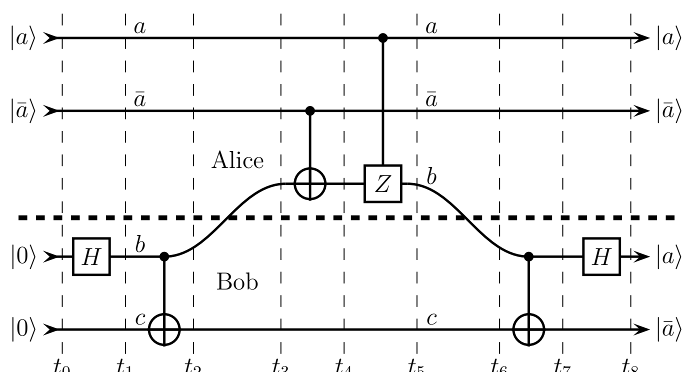
form an orthonormal basis of fully entangled states on the tensor product $\mathcal{H}_a \otimes \mathcal{H}_b$ of two two-dimensional Hilbert spaces, where each space has an orthonormal basis $\{|0\rangle, |1\rangle\}$.

(a) Find unitary operations of the form $U_a \otimes I_b$ that map $|B^0\rangle$ to $|B^1\rangle$, $|B^2\rangle$, and $|B_3\rangle$. Do you recognize the U_a operators?

(b) Suppose Alice and Bob share the Bell state $|B^j\rangle$ for some (unknown j. That is, Alice holds bit a and Bob holds bit b. How many bits of classical information are required to specify the value of j?

(c) What can Alice learn about j by measuring the state of her bit ($|a\rangle = |0\rangle$ or $|1\rangle$)? If Bob measures the state of his bit $|b\rangle$ and shares his knowledge with Alice, what can they jointly learn about j?

(d) Now let Alice and Bob each have two bits, $|a\rangle$ and $|\bar{a}\rangle$, and $|b\rangle$ and $|c\rangle$, respectively as shown in the figure. Show that the state at time 2 is a product state, $|\Psi_2\rangle = |a\rangle|\bar{a}\rangle|B^j\rangle$, and determine the value of j.



(e) If the state at time 6 took the form $|\Psi_6\rangle = |a\rangle|\bar{a}\rangle|B^k\rangle$, then what would the state $|\Psi_8\rangle$ be? Work this out for each $k \in 0, \ldots, 3$.

(f) Show that Alice can transfer two classical bits, a and $\bar{a}$, to Bob by passing him the single qubit $|b\rangle$.

2. Let two operators A and B satisfy the commutation relation $[B, A] = iI$. Explain why the Hilbert space must be infinite dimensional.