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33-756 Midterm Exam 1 Monday, March 3, 2014

Cohen-Tannoudji, Chapter 11 Complement B.

a. Consider a pair of spin-1/2 particles placed at positions $= \pm a\hat{z}$ in an external magnetic field $\mathbf{B_0} = B_0\hat{z}$. Neglecting other interactions, the Hamiltonian for the system is

$$H_0 = \omega_1 S_{1z} + \omega_2 S_{2z},$$

where the frequencies $\omega_i = -\gamma_i B_0$ with γ_i the gyromagnetic ratio of particle i, and S_{iz} is the z-compoment of the spin operator acting on particle i. Suppose $0 < \omega_1 < \omega_2$.

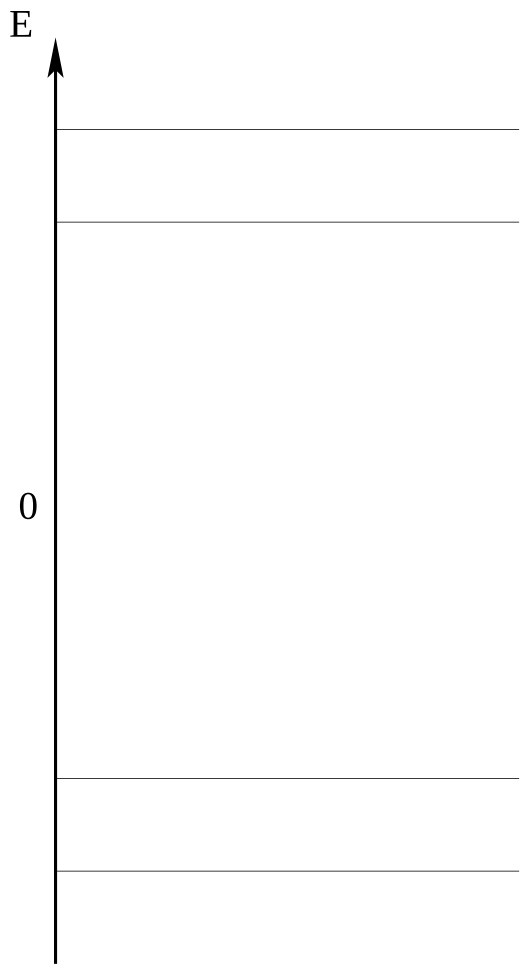
You may wish to recall that for a spin-1/2 particle, $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$. Hence the eigenstates of H_0 can be denoted $|\sigma_1\sigma_2\rangle$, where $\sigma_i = +$ or $-$. The energy eigenvalue of the state $|\sigma_1\sigma_2\rangle$ is

$$E_{\sigma_1\sigma_2} = \frac{\hbar}{2}\big(\sigma_1\omega_1 + \sigma_2\omega_2\big)$$

as illustrated in the diagram on the following page.

(i) Determine which pairs of energy eigenstates exhibit nonzero matrix elements of the x-component of total magnetic moment $M_x \equiv \gamma_1 S_{1x} + \gamma_2 S_{2x}$. It may help to recall that the raising and lowering operators $S_\pm = S_x \pm iS_y$ obey $S_\pm|\pm\rangle = 0$ and $S_\pm|\mp\rangle = \hbar|\pm\rangle$. Give a complete list of all pairs $|\alpha\rangle$ and $|\beta\rangle$ for which $\langle\alpha|M_x|\beta\rangle \neq 0$, and indicate these on the diagram below using arrows to denote allowed transitions.

These energy level differences, divided by $\hbar$, are the Bohr frequencies that are measured in NMR experiments with the radio frequency magnetic field polarized along the x axis.



(ii) On the axes below sketch the expected NMR signal and label all resonant frequencies.

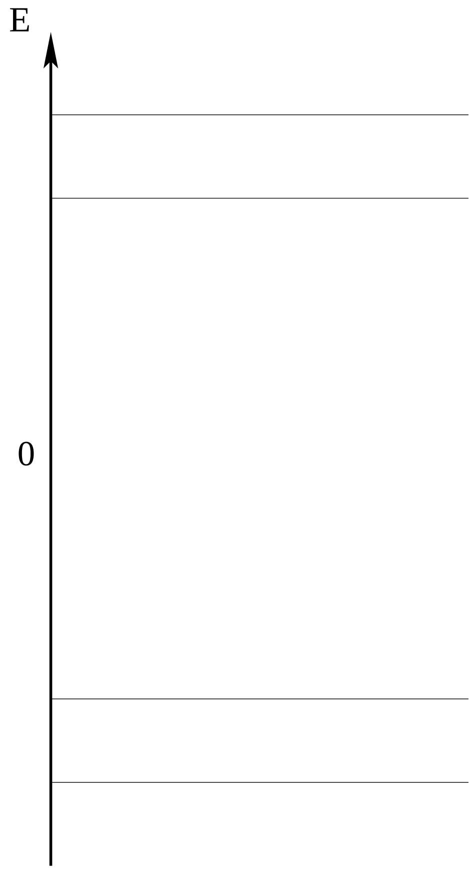
b. The dipole-dipole interaction of the particles, whose separation is parallel to the z-axis, is

$$W = \frac{\Omega}{\hbar}(4S_{1z}S_{2z} - S_{1+}S_{2-} - S_{1-}S_{2+}),$$

is sufficiently small that $\Omega << \omega_2 - \omega_1$ and W can be considered as a perturbation on H_0.

(i) Calculate the shifted energy levels to first order in perturbation theory.

(ii) Next to the unperturbed energy level diagram indicate the level shifts caused by the perturbation.



(iii) Sketch the expected NMR signal on the axes below and label all resonant frequencies.

c. Consider the case where the two particles are equivalent, for example a pair of protons in a water molecule, so that $\gamma_1 = \gamma_2$ and hence $\omega_1 = \omega_2 \equiv \omega$.

(i) Calculate the energy levels to first order in perturbation theory.

(ii) Sketch and label the unperturbed energy level diagram and indicate the energy level shifts caused by the perturbation W. Indicate which pairs of eigenstates exhibit nonzero matrix elements of M_x.

(iii) Sketch the expected NMR signal on the axes below and label all resonant frequencies.